Time space tradeoffs for attacks against one-way functions and PRGs

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Joint work with
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- Are better (non-uniform) attacks possible against:
  - one-way functions?
  - pseudo-random generators?

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- PRG: $G : [N] \to [2N]$ is a (t, ϵ)-secure PRG if for every algorithm A of complexity $\leq t$

$$\left| \Pr_{x \sim [N]} [A^G(G(x)) = 1] - \Pr_{y \sim [2N]} [A^G(y) = 1] \right| \leq \epsilon$$

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- Can be implemented on a RAM machine with time and space t.
- Similar to circuit complexity.

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All above results are actually stated as time-space tradeoffs. Complexity is optimized when $T = S$.

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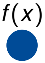
Hellman's approach for permutations



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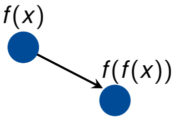

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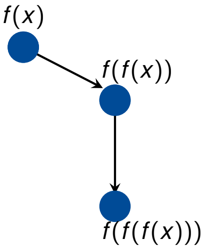

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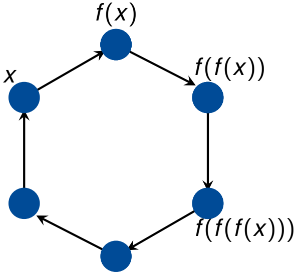

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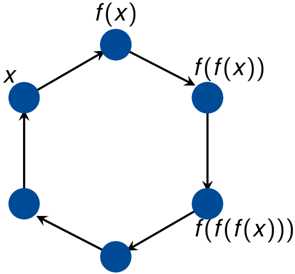

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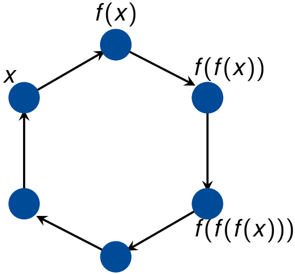

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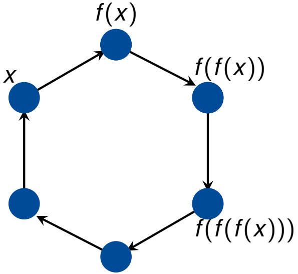

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Time complexity of computation is $\tilde{O}(\sqrt{N})$.

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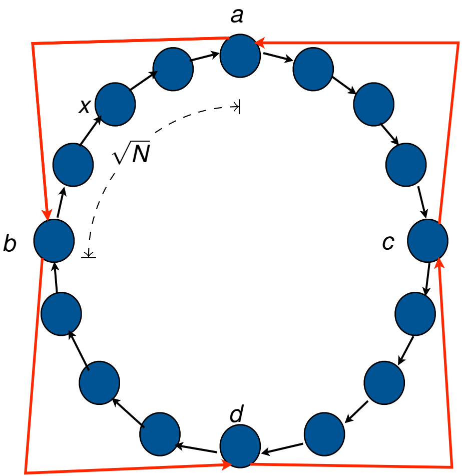

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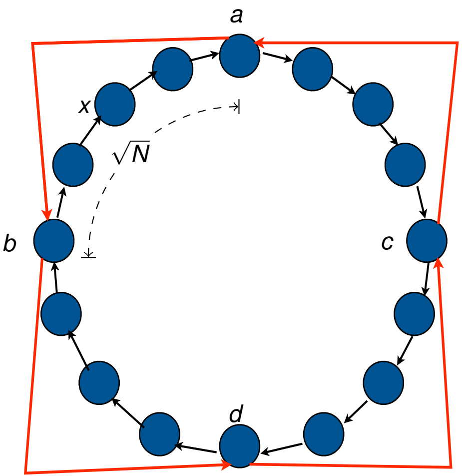

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For e.g., store (a, b), (b, c), (c, d) and (d, a) in a data-structure

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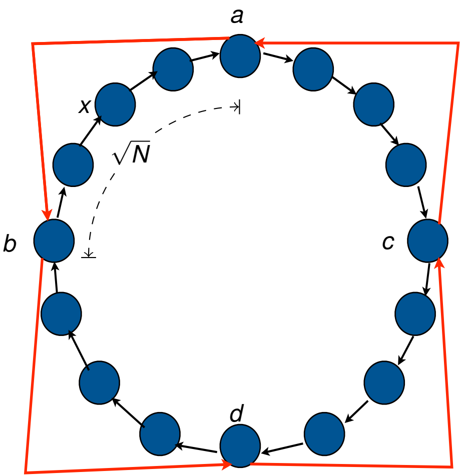

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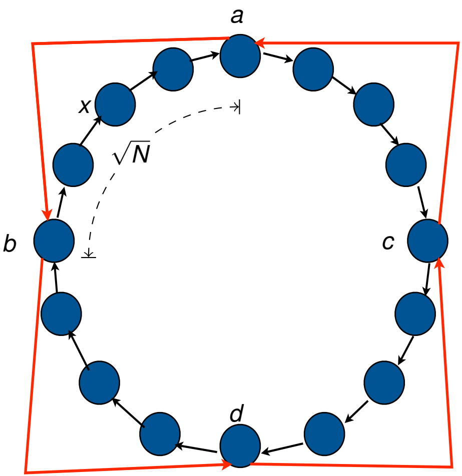

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When you hit a, use back-link to go back to b

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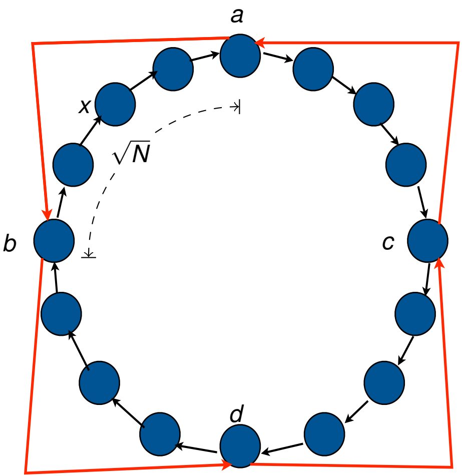

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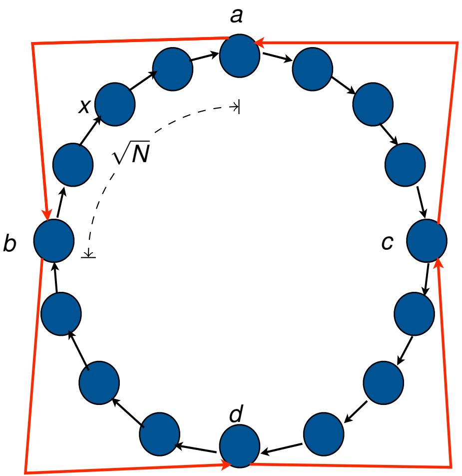

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The penultimate point in the sequence is $f^{-1}(x)$

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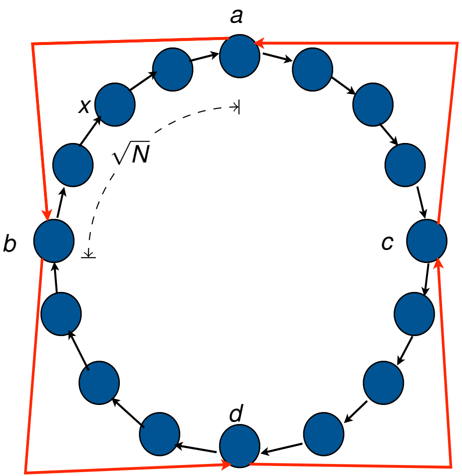

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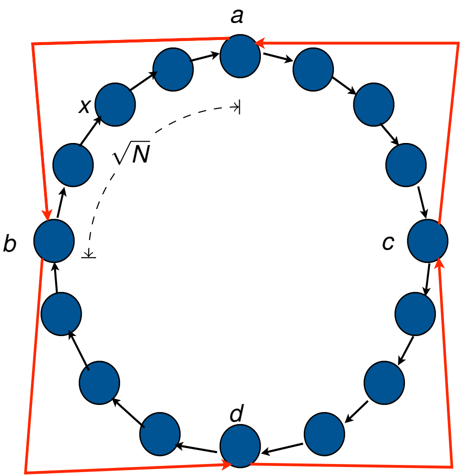

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Also, the total time complexity is $\sqrt{N}$ as you hit a “back-link” in that time

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- Total time $T = \tilde{O}(\sqrt{N})$ and space $S = \tilde{O}(\sqrt{N})$.
- Can be used to invert ϵ fraction of the elements in time $T = \tilde{O}(\sqrt{\epsilon N})$ and space $S = \tilde{O}(\sqrt{\epsilon N})$
- In fact, we can achieve any time (T) space (S) tradeoff such that $T \cdot S = \epsilon N$.

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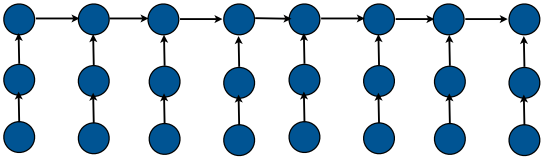
- Problem: m may have to be very large because the graph $(x \to f(x))$ may not have many long and disjoint paths.



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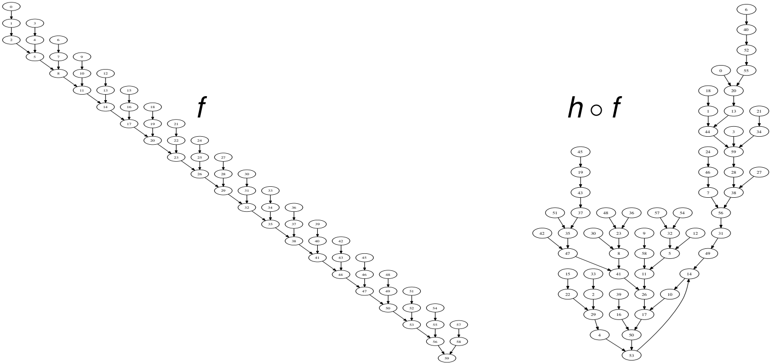

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- Choose $m, \ell, r = \tilde{O}(N^{1/3})$.

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- Problems: Computing $h_1, \ldots, h_r$ is hard. Heuristic works only for random f.

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- Can again choose m, l such that $m\ell^2\lambda \approx m\ell^2/K \ll 1$. Can get

$$T, S = \tilde{O}(N^{3/4})$$

by taking $K = \tilde{O}(N^{3/4})$, $r = \tilde{O}(N^{1/2})$ and $m, \ell = \tilde{O}(N^{1/4})$.

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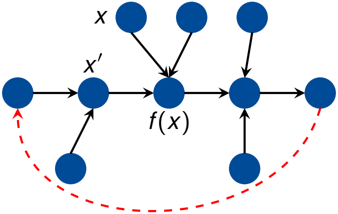
- First observation: If a table of size K does not invert f with probability ϵ, then the collision probability for the rest is ϵ/K.
- Second Observation: The number of elements inverted by a path is not just the path length, but the the sum of indegrees of elements in the path.



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- Final complexity: $T, S \;=\; \begin{array}{ll} \tilde{O}(\sqrt{\epsilon N}) & \epsilon \leq N^{-1/3} \\ \tilde{O}(\epsilon^{5/4} N^{3/4}) & \epsilon \geq N^{-1/3} \end{array}$

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- As in [GT00], show that using A, can encode f with $\approx \log(N!) - \phi(N, T) + S$ bits for some ϕ. Thus, $S > \phi(N, T)$ giving the tradeoff between T and S.

Lower bound for inverting permutations

- Given A inverting f on ϵ fraction of inputs in time T and space S, want to show $T \cdot S = \Omega(\epsilon N)$.
- Showed by [Yao90] for $\epsilon = 1$ and [GT00], [Wee05] when $T = O(\sqrt{\epsilon N})$.
- Give a simpler, "randomized" proof that works for all T. Also extends to lower bounds for PRGs.
- As in [GT00], show that using A, can encode f with $\approx \log(N!) - \phi(N, T) + S$ bits for some ϕ. Thus, $S > \phi(N, T)$ giving the tradeoff between T and S.
- We show that using A, one can encode f using $\approx \log(N!) - \frac{\epsilon N}{100T} + S$ bits giving us the desired tradeoff.

Intuition for the encoding

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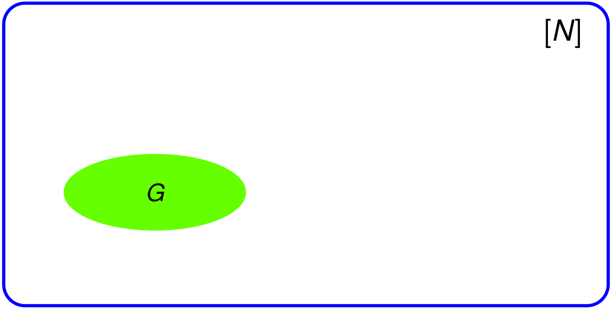

$$|G| = \frac{\epsilon N}{100T}$$

- A inverts G correctly.
- For all $x \in G$, A does not query any element in G.

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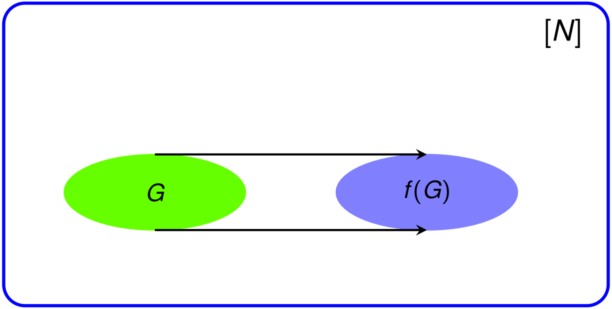

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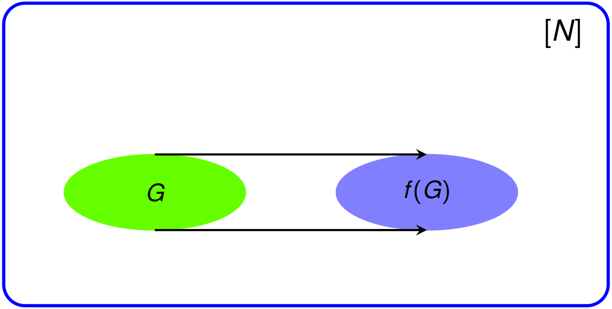

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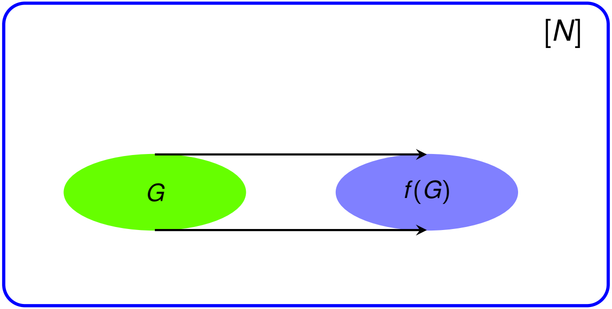

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- Complexity of encoding :=
  - Size of G
  - Specify set $f(G)$
  - Specify the map f^{-1} on $[N] - f(G)$

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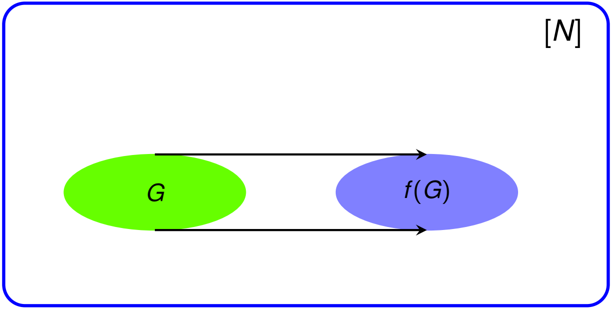
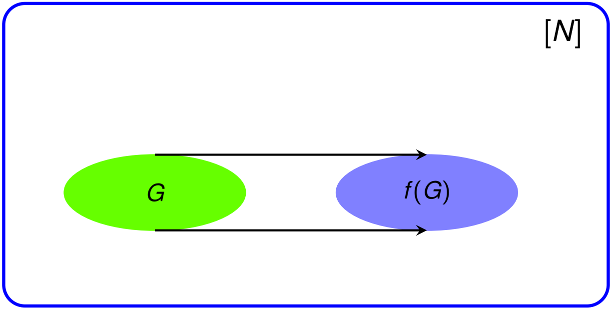


$$|G| = \frac{\epsilon N}{100T}$$

- Complexity of encoding :=
  - Size of G
  - Specify set $f(G)$
  - Specify the map f^{-1} on $[N] - f(G)$
- This information along with A suffices to specify f entirely

Intuition for the encoding



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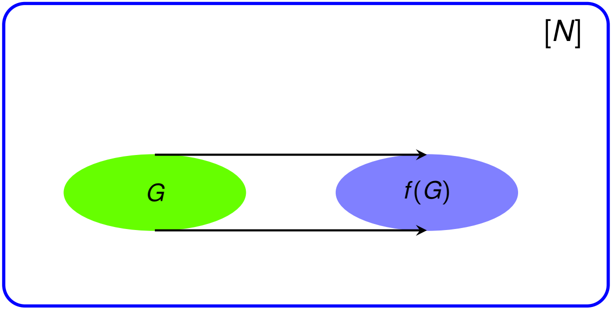

$$|G| = \frac{\epsilon N}{100T}$$

- Total complexity of encoding : $2\log\binom{N}{|G|} + \log(N - |G|)!$
- Putting $|G| = \frac{\epsilon N}{100T}$, we get that $S + \frac{\epsilon N}{T}\log(T^2/\epsilon^2 N) \geq 0$

Upshot of the analysis

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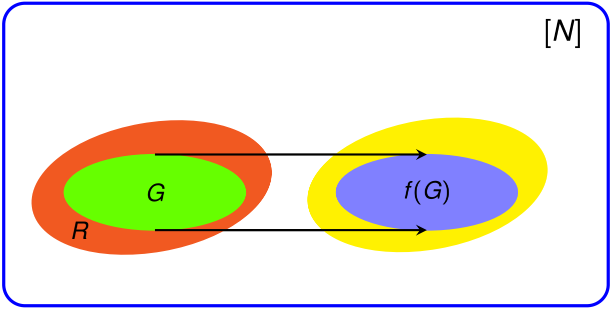
- This was the analysis by Gennaro and Trevisan [GT00]
- The analysis was improved by Wee [Wee05] who showed $TS = \tilde{\Omega}(\epsilon N)$ provided $T \leq \sqrt{\epsilon N}$
- There is still a gap because “deterministically” deciding on G is very expensive.

Randomized encoding



$$|G| = \frac{\epsilon N}{100T}$$

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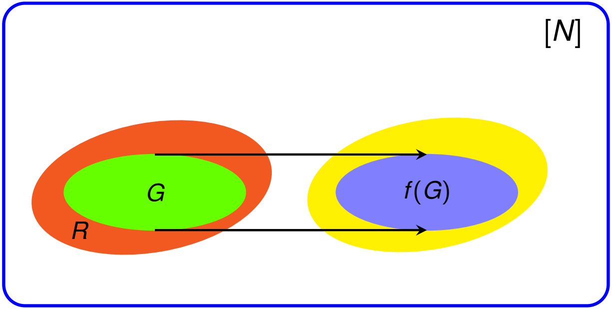

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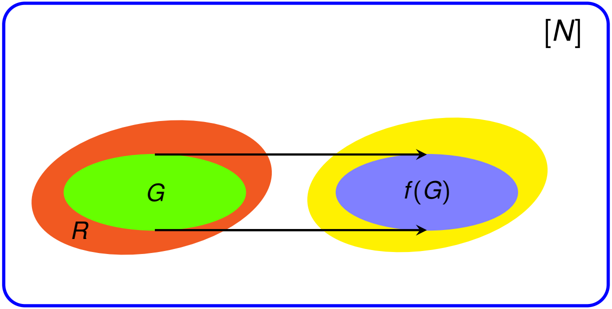

$$|G| = \frac{\epsilon N}{100T}$$

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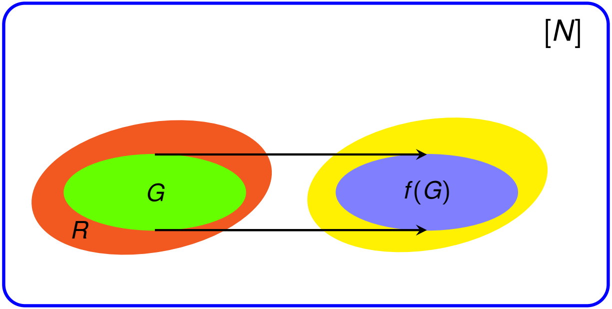
- Choose R to be a set of size $N/10T$ uniformly at random.
- With high probability, this contains a set G of size $\frac{\epsilon N}{100T}$ such that
    - A inverts G correctly.
    - For all $x \in G$, A does not query any element in R

Randomized encoding



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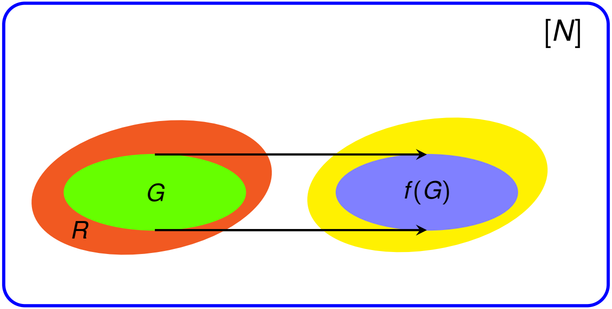

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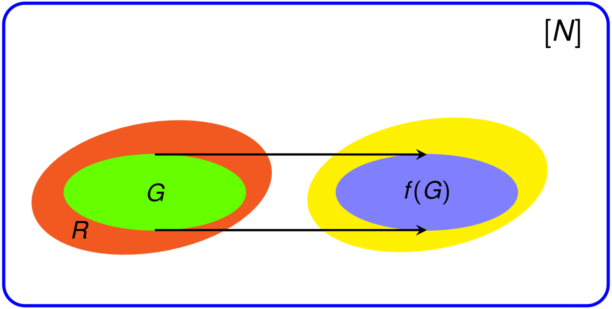

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- Some savings in the analysis as the identity of R is already known
- Once we know f outside R, we need to know "G in R" as opposed to "G in [N]" - main source of saving
- In all, we can describe the permutation in $\log(N!) - \epsilon N/100T + S$ bits which gives us the result.

Conclusions

- Non-uniform attacks can do better than uniform attacks on one-way functions and PRGs

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Conclusions

- Non-uniform attacks can do better than uniform attacks on one-way functions and PRGs
- The best provable upper bound for one-way functions on all inputs remains $N^{3/4}$ and $N^{2/3}$ is the best for "Hellman"-style arguments (Barkan, Biham and Shamir)
- Techniques for proving lower bounds do not seem to do any better for one-way functions than permutations i.e. $\Omega(N^{1/2})$.

Thank You

Questions?