The Lovász Local Lemma and Satisfiability

Algorithmic Aspects

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1. Introduction
   - Problem overview
2. Proof
   - Algorithm
   - The proof
3. Further work

Setting: the k-SAT problem

a k-CNF formula:

$$F = \underbrace{(x_1 \vee \bar{x}_2 \vee \ldots \vee \bar{x}_{19})}_{k} \wedge \underbrace{(\bar{x}_3 \vee x_7 \vee \ldots \vee x_{19})}_{k} \wedge \ldots$$

A conjunction of disjunctions (*clauses*), each composed of k *literals*.

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n : number of variables
m : number of clauses
$\mathrm{vbl}(C)$: the set of variables occurring in clause C

A simple subclass

Neighbourhood of a clause $C \in F$:

$$\Gamma(C) := \{\ D \in F \mid D \neq C, \mathrm{vbl}(C) \cap \mathrm{vbl}(D) \neq \emptyset\ \}$$

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- Q: can we *find* a satisfying assignment?

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Previous approaches to the problem:

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Theorem

There exists a randomized algorithm which, given a k-CNF formula F with $\forall C \in F : |\Gamma^+(C)| \leq 2^{k-3}$, finds a satisfying assignment for F in expected polynomial time.

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      while(∃C ∈ F : C violated)
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solve(F):
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      while(∃C ∈ F : C violated) // repeats <= m times
            pick lexicographically first such C
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locally_correct(C):
      resample new values for vbl(C)
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   log(<-)
```

Logging the Program Execution

A sample log looks like this [with storage space requirements]:

`new recursion for 6`	[$\log m$ bits]
`    next clause 1`	[$(k-3)+1+1$ bits]
`        next clause 2`	[$(k-3)+1+1$ bits]
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`    ...`	

Lemma

Each line of the log allows to reconstruct k bits of the random input used by the algorihtm.

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The process has to stop in $\mathcal{O}(m \log m)$ time. □

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  - $2^k/e$ neighbours (no gap anymore)
  - rules on choice of constraints not necessary
  - generalization to applications beyond SAT

Derandomization and Output Distribution

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- Haeupler, Saha, Srinivasan '10:
  - analyse the output distribution of the algorithm
  - settings with super-polynomially many events
  - interesting applications (e.g. Santa Claus problem and coloring problems)

Simplified algorithm

The more sophisticated proof variant demonstrates the following simplified algorithm to terminte in $\mathcal{O}(mk)$ expected time:

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solve(F):
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    Schöning: flip one u.a.r. from vbl(C)
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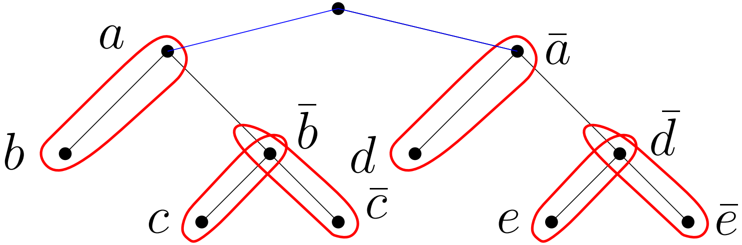
Open questions:

- does Schöning work for the LLL case?
- is there such a jump in complexity or can it be smoothened?

Structural results on the 'jump'

By Gebauer, Szabó and Tardos: there are unsatisfiable formulas where each clause has $2^k(e^{-1} + o(1))$ neighbors, so the LLL is asymptotically tight for SAT.



The formulas can be constructed in such a way that no variables is featured in more than $(2/e + \epsilon) \cdot 2^k/k$ clauses.

Thanks

THANK YOU