A Unified Continuous Greedy Algorithm for Submodular Maximization

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Submodular Functions

Ground set $\mathcal{N}$ and $f : 2^{\mathcal{N}} \to \mathbb{R}$.

Submodular - Definition

$$f(A \cup B) + f(A \cap B) \leqslant f(A) + f(B) \quad \forall A, B \subseteq \mathcal{N}.$$

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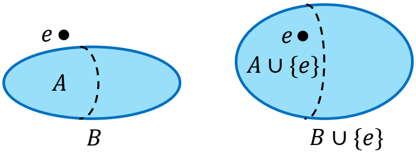
Decreasing Marginal Values:



$$f(A \cup \{e\}) - f(A) \geqslant f(B \cup \{e\}) - f(B)$$
$$\forall A \subseteq B \subseteq \mathcal{N}, \forall e \in \mathcal{N} \setminus B.$$

Submodular Functions (Cont.)

Monotonicity - Definition

$$f(A) \leqslant f(B) \quad \forall A \subseteq B \subseteq \mathcal{N}.$$

Submodular Functions (Cont.)

Common in combinatorial optimization:

1. Rank functions of matroids.
2. Cuts in undirected and directed graphs.
3. Cuts in hypergraphs.
4. Covering functions.

Abundant uses in game-theory and economics.

Submodular Maximization

Optimization Problem

Family of allowed subsets $\mathcal{M} \subseteq 2^{\mathcal{N}}$.

$$\begin{aligned} \max \quad & f(S) \\ s.t. \quad & S \in \mathcal{M} \end{aligned}$$

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Value Oracle Model: Returns $f(S)$ for given $S \subseteq \mathcal{N}$.

Submodular Maximization - Problems

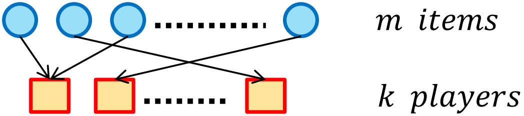
Submodular Welfare

- k players and m items $\mathcal{Q}$.
- Monotone and submodular $f_i : 2^{\mathcal{Q}} \to \mathbb{R}^+$ for player i.

Goal: allocate each item to a single player to maximize

$$\sum_{i=1}^{k} f_i(\mathcal{Q}_i).$$

$\mathcal{Q}_i$ - items allocated to player i



Submodular Maximization - Problems (Cont.)

- $\max\left\{1-\frac{1}{e}, \frac{2}{2k-1}\right\}$-approximation [Calinescu-Chekuri-Pal-Vondrák], [Dobzinski-Schapira], [Vondrák].
- $\left(1-\left(1-\frac{1}{k}\right)^k\right)$-hardness [Vondrák].

Additional work:

- Variants: [Dobzinski-Nisan-Schapira], [Feige-Vondrák], [Feige].
- Hardness: [Chakrabarty-Goel], [Khot-Lipton-Markakis-Mehta].

Submodular Maximization - Problems (Cont.)

Generalization of Submodular Welfare with 2 players.

Submodular Max-SAT

- CNF formula with n variables and m clauses $\mathcal{Q}$.
- Monotone submodular $f : 2^{\mathcal{Q}} \to \mathbb{R}^+$ over the clauses.

Goal: find an assignment which satisfies clauses $S \subseteq \mathcal{Q}$ that maximizes $f(S)$.

Submodular Maximization - Problems (Cont.)

Generalization of Submodular Welfare with 2 players.

Submodular Max-SAT

- CNF formula with n variables and m clauses Q.
- Monotone submodular $f : 2^Q \to \mathbb{R}^+$ over the clauses.

Goal: find an assignment which satisfies clauses $S \subseteq Q$ that maximizes $f(S)$.

Related work:

- $(2/3)$-approximation [Azar-Gamzu-Roth], [Dobzinski-Schapira].
- $(3/4)$-hardness [Vondrák].

Submodular Maximization - Problems (Cont.)

A Matroid Constraint

- Matroid $M = (\mathcal{N}, \mathcal{I})$.
- Submodular $f : 2^{\mathcal{N}} \to \mathbb{R}^+$ - not monotone !

Goal: find an independent set $S \in M$ that maximizes $f(S)$.

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Related work:

- ≈ 0.325-approximation [Ageev-Sviridenko], [Chekuri-Vondrák-Zenklusen], [Gharan-Vondrák].
- Hardness: ≈ 0.478 [Gharan-Vondrák].

Submodular Maximization - Problems (Cont.)

$O(1)$ Knapsack Constraints

- $O(1)$ knapsack constraints on n elements $\mathcal{N}$.
- Submodular $f : 2^{\mathcal{N}} \to \mathbb{R}^+$ - not monotone !

Goal: find a feasible packing $S \subseteq \mathcal{N}$ that maximizes $f(S)$.

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Related work:

- ≈ 0.325-approximation [Chekuri-Vondrák-Zenklusen], [Kulik-Shachnai], [Lee-Mirrokni-Nagarajan-Sviridenko].
- $1/2$-hardness can be derived from [Vondrák].

Approximating a Submodular Maximization Problem

Combinatorial Approach:

- Local search, greedy rules etc.
- Used as early as the late 70's.
- Provides current state of the art and tight
  [Feldman-Naor-S-Ward], [Lee, Sviridenko, Vondrák], [Sviridenko].
- Usually tailored for a specific structure.

Approximating a Submodular Maximization Problem

Continuous Approach:

1. Formulate a relaxation.
2. Find a *good* fractional solution.
3. Round solution.

Notable example: *asymptotically tight* approximation for a monotone submodular function over a matroid [Calinescu, Chekuri, Pal, Vondrák], [Nemhauser, Wolsey], [Nemhauser, Wolsey, Fisher].

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Note

Improved fractional solution $\Rightarrow$ improved approximation !

Our Results

Main Result

A simple algorithm that finds better fractional solutions.

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Some Applications:

- Tight results for Submodular Welfare and Submodular Max-SAT.
- Improved $(1/e)$-approximation for maximizing a non-monotone submodular function given a matroid or $O(1)$ knapsack constraints.

Formulating a Relaxation

How to formulate a relaxation ?

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Multilinear Relaxation

- $\mathcal{P} \subseteq [0,1]^{\mathcal{N}}$ - polytope containing $\mathcal{M}$.
- $F(x) = \sum_{R \subseteq \mathcal{N}} f(R) \prod_{e \in R} x_e \prod_{e \notin R}(1 - x_e)$, $\forall x \in [0,1]^{\mathcal{N}}$.

$$\begin{array}{rl} \max & F(x) \\ s.t. & x \in \mathcal{P} \end{array}$$

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$$\begin{array}{rl} \max & F(x) \\ s.t. & x \in \mathcal{P} \end{array}$$

Problem

F is neither convex nor concave - **how to solve relaxation?**

Solving Multilinear Relaxations - Monotone f

- Works only for monotone f.
- $\mathcal{P}$ is down-monotone and solvable.

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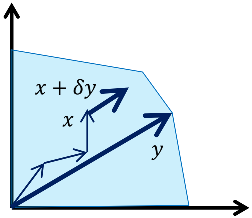
Continuous Greedy Algorithm

1. $x \leftarrow 0$, $t \leftarrow 0$.
2. While $t < 1$:
   1. $w_e = F(x \vee \mathbf{1}_e) - F(x)$, $\forall e \in \mathcal{N}$.
   2. $y = \text{argmax}\{w \cdot y \mid y \in \mathcal{P}\}$.
   3. $x \leftarrow x + \delta \cdot y$.
   4. $t \leftarrow t + \delta$.
3. Output x.



Solving Multilinear Relaxations - Monotone f (Cont.)

- First suggested by [Calinescu-Chekuri-Pal-Vondrák], [Vondrák].
- Simple and elegant.
- Achieves asymptotically tight results in some cases.
- Outputs a convex combination of points in $\mathcal{P}$.

Theorem

The continuous greedy algorithm finds a feasible solution with value at least $(1 - e^{-t})f(OPT)$ when terminated at time $t \in [0, 1]$.

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Question

What happens when f is not necessarily monotone ?

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What happens when f is not necessarily monotone ?

- The algorithm fails !
- Much more involved methods are known - the best achieves an approximation of ≈ 0.325 [Chekuri-Vondrák-Zenklusen].

Intuition

Improvement determined by **residual increase**:

$$w_e \triangleq F(x \vee \mathbf{1}_e) - F(x).$$

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Monotonicity of f is used when showing:

$$w_e \leqslant \partial_e F(x).$$

Can we do better?

Intuition (Cont.)

Observation

$$w_e \triangleq F(x \vee \mathbf{1}_e) - F(x) = \partial_e F(x) \cdot (1 - x_e)$$

$$\Downarrow$$

Change update step to: $x_e \leftarrow x_e + \delta \cdot y_e \cdot (1 - x_e)$, $\forall e \in \mathcal{N}$.

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Unified Continuous Greedy Algorithm

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   3. $x_e \leftarrow x_e + \delta \cdot y_e \cdot \mathbf{(1 - x_e)}$, $\forall e \in \mathcal{N}$.
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A unified continuous greedy algorithm for both cases.

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Theorem [Feldman-Naor-S]

The unified continuous greedy algorithm finds a fractional solution with value at least:

1. $(1 - e^{-t})f(OPT)$ (f is monotone),
2. $(t \cdot e^{-t})f(OPT)$ (general f),

when terminated at time t.

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Monotone f - choice of t depends on $\mathcal{P}$

- **density** of $\mathcal{P}$ determines best $t \geqslant 1$
- t might be larger than 1 and still $x \in \mathcal{P}$.
  (original continuous greedy is somewhat wasteful)

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- At time t:

$$\sum_{e \in \mathcal{N}} y_e \cdot (1 - x_e) \cdot \partial_e(F(x)) \geqslant F(x \vee \mathbf{1}_{OPT}) - F(x).$$

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$$F(x \vee \mathbf{1}_S) \geqslant (1 - a) f(S).$$

- Bound the value of x_e at each step.
- Find a recursive function bounding the improvement in $F(x)$ at every step.

Open Questions

1. Is the $(1/e)$-approximation for non-monotone f tight?
2. Are there additional applications for the unified continuous greedy algorithm?
   (Unconstrained Submodular Maximization)

Thank You !