Exponential Time Algorithms

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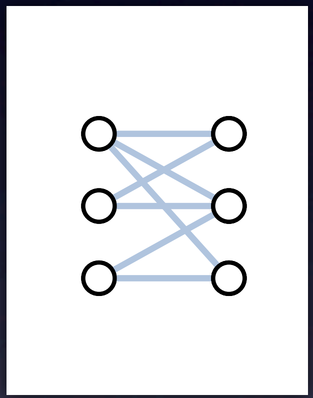
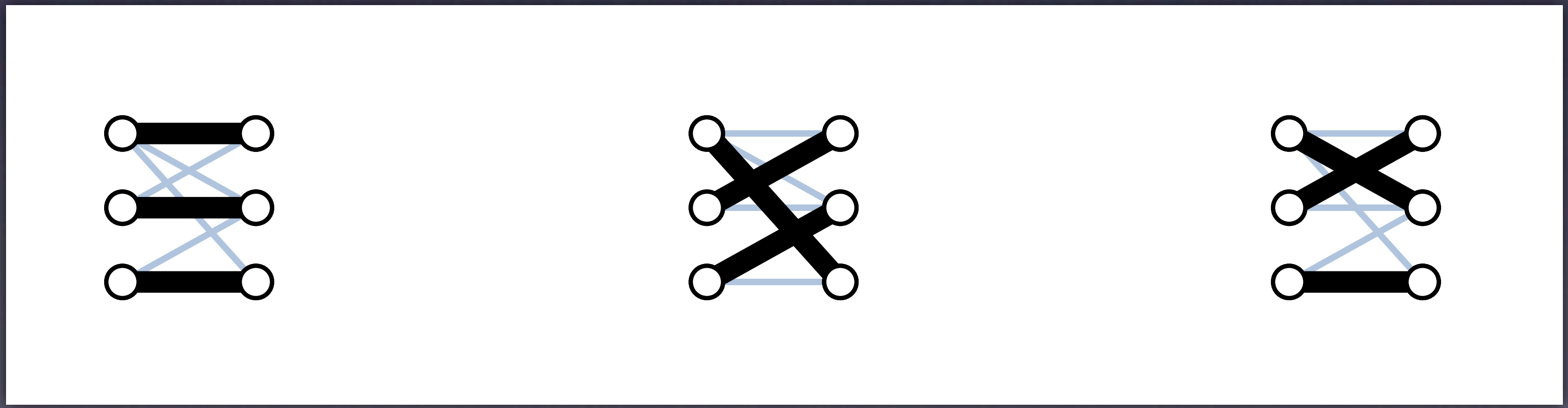
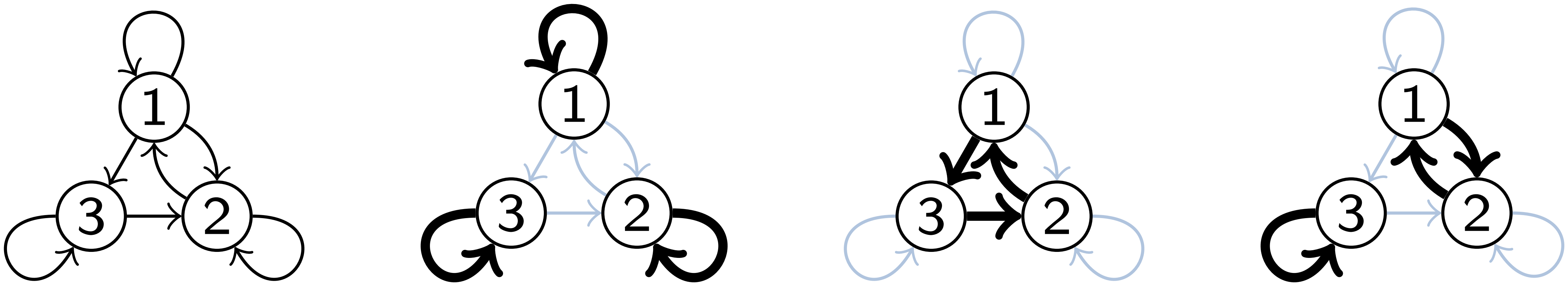
Perfect Matchings

Perfect Matchings in Bipartite Graphs

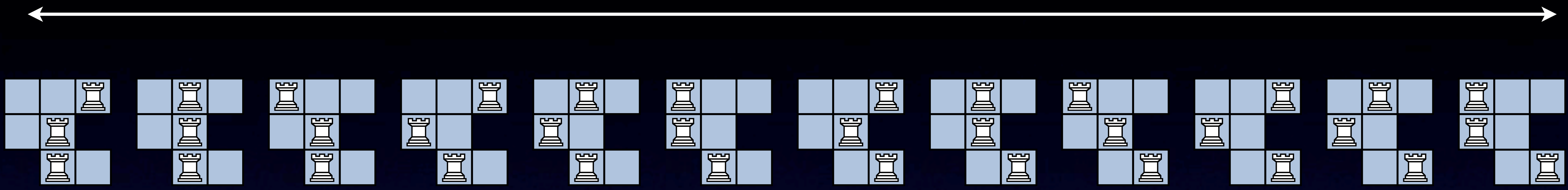




$$\begin{matrix}1&1&1\\1&1&0\\0&1&1\end{matrix}\qquad\begin{matrix}1&1&1\\1&1&0\\0&1&1\end{matrix}\qquad\begin{matrix}1&1&1\\1&1&0\\0&1&1\end{matrix}\qquad\begin{matrix}1&1&1\\1&1&0\\0&1&1\end{matrix}$$



All ways of placing 1 rook per row



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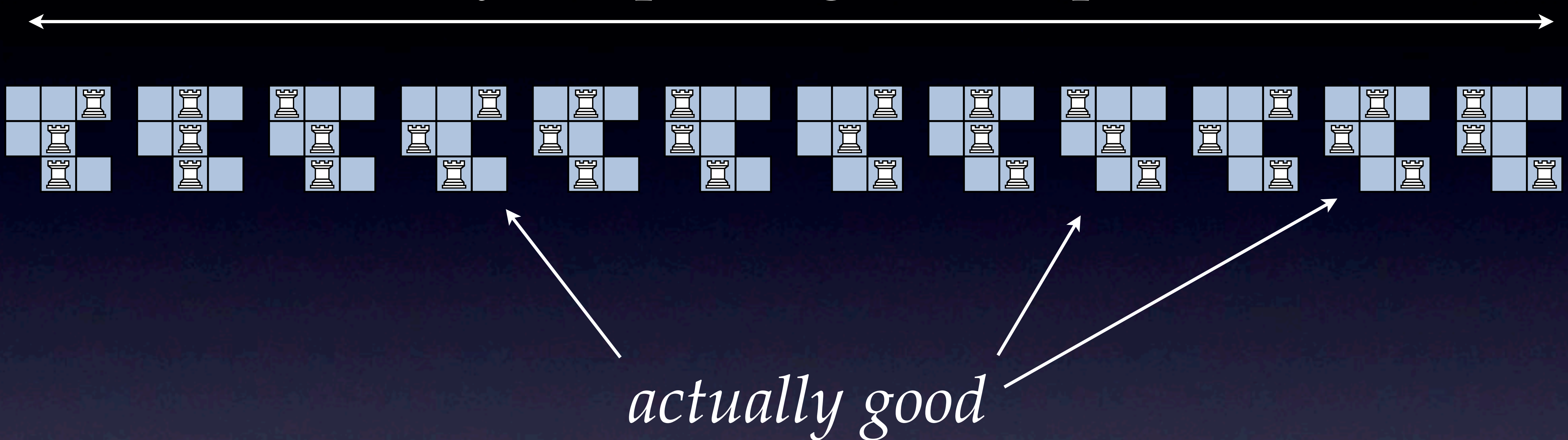

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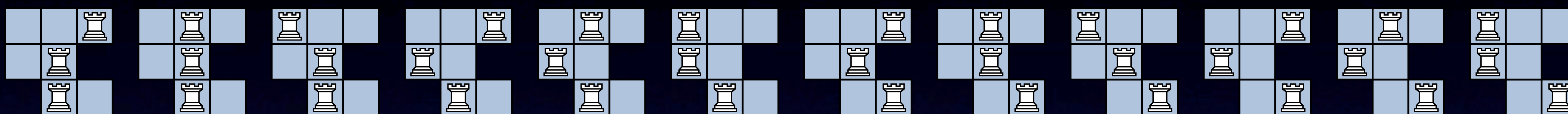

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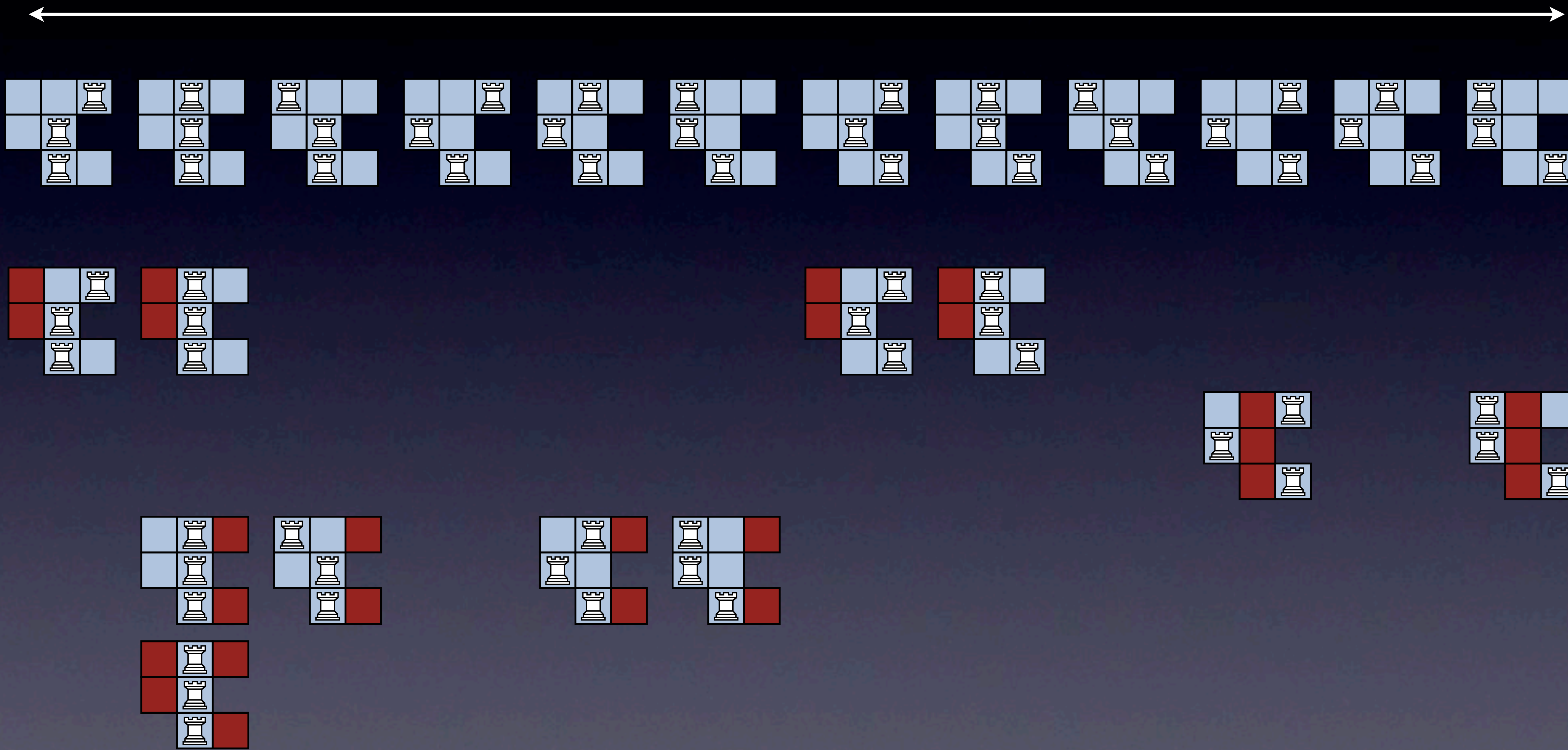
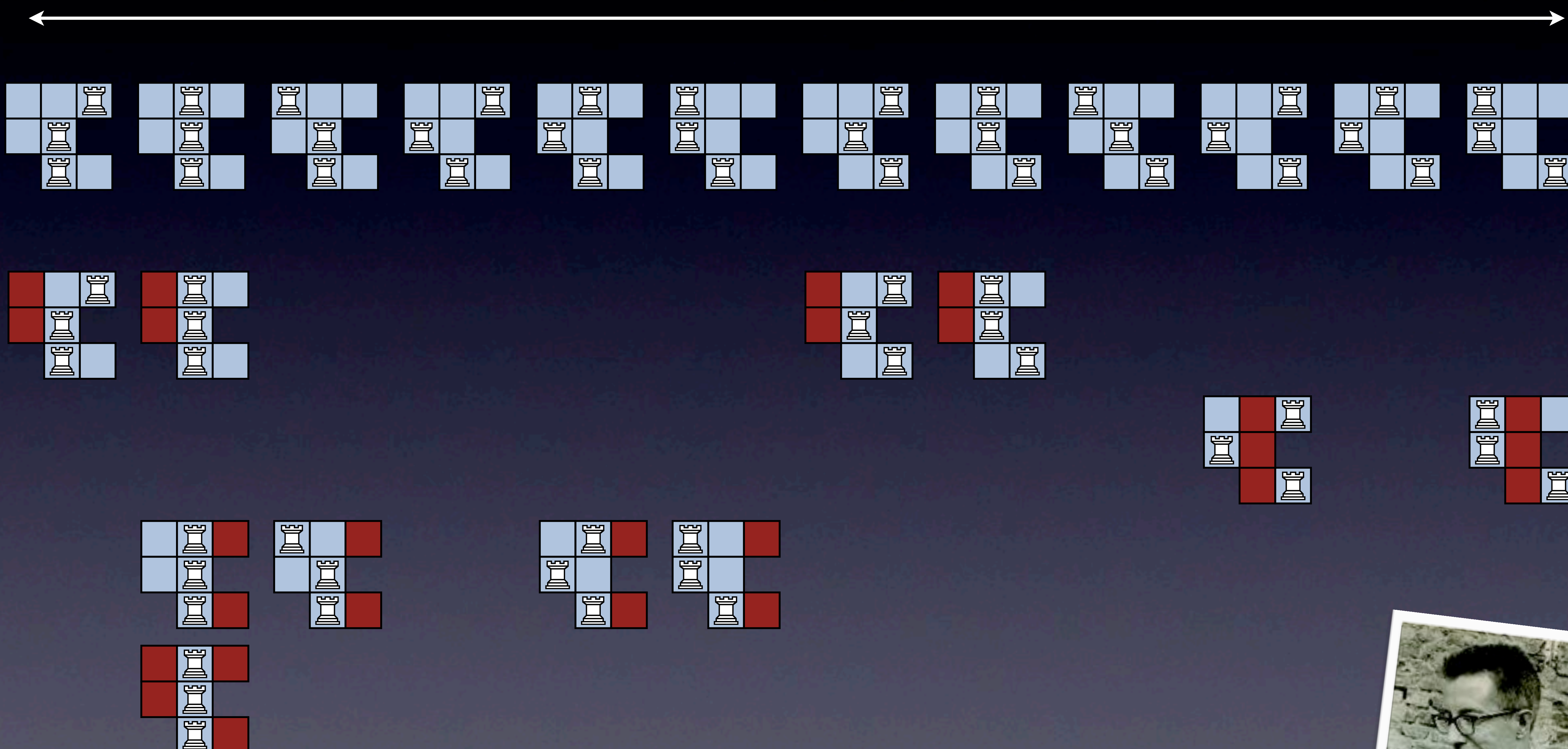
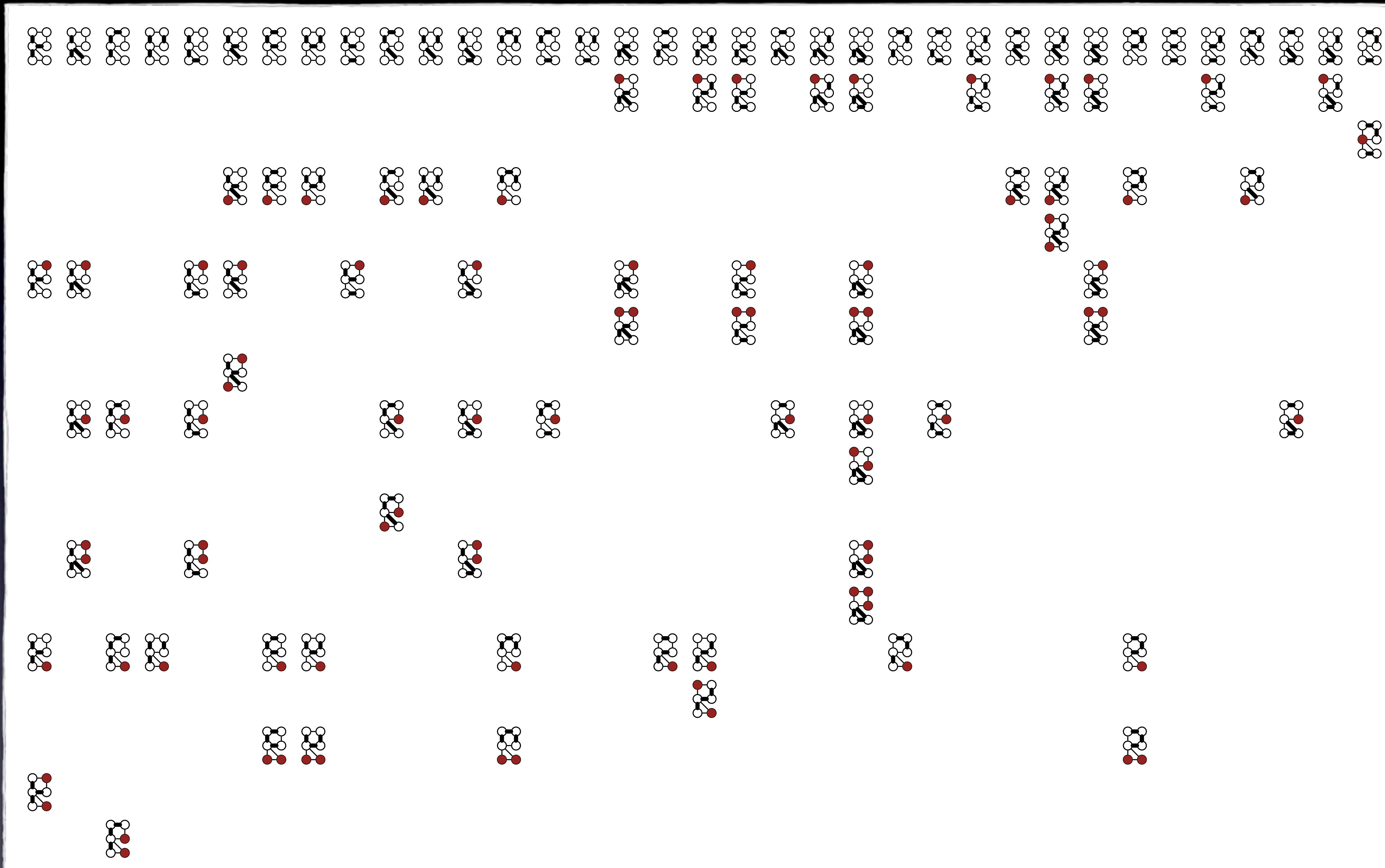
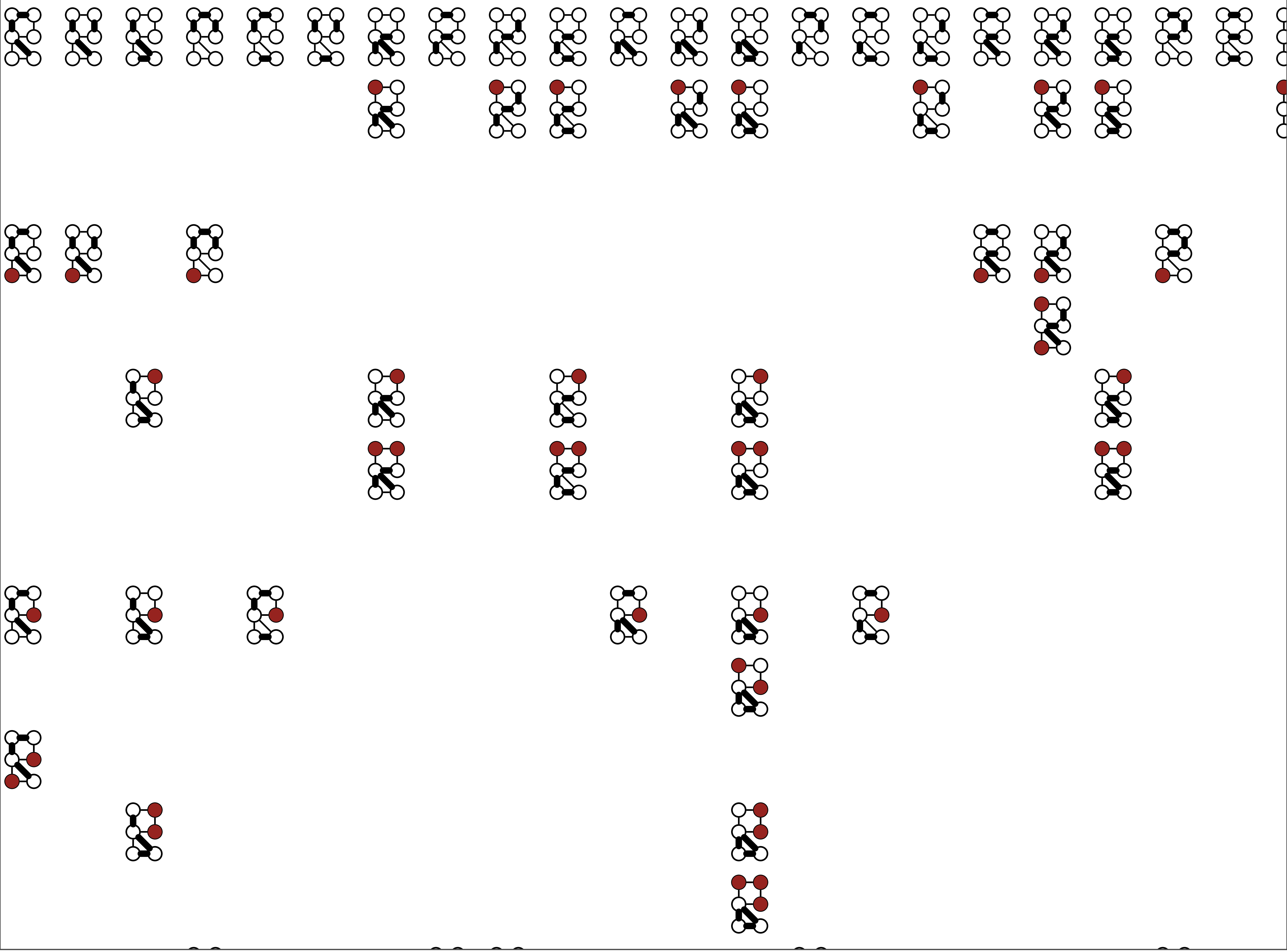


All ways of placing 1 rook per row

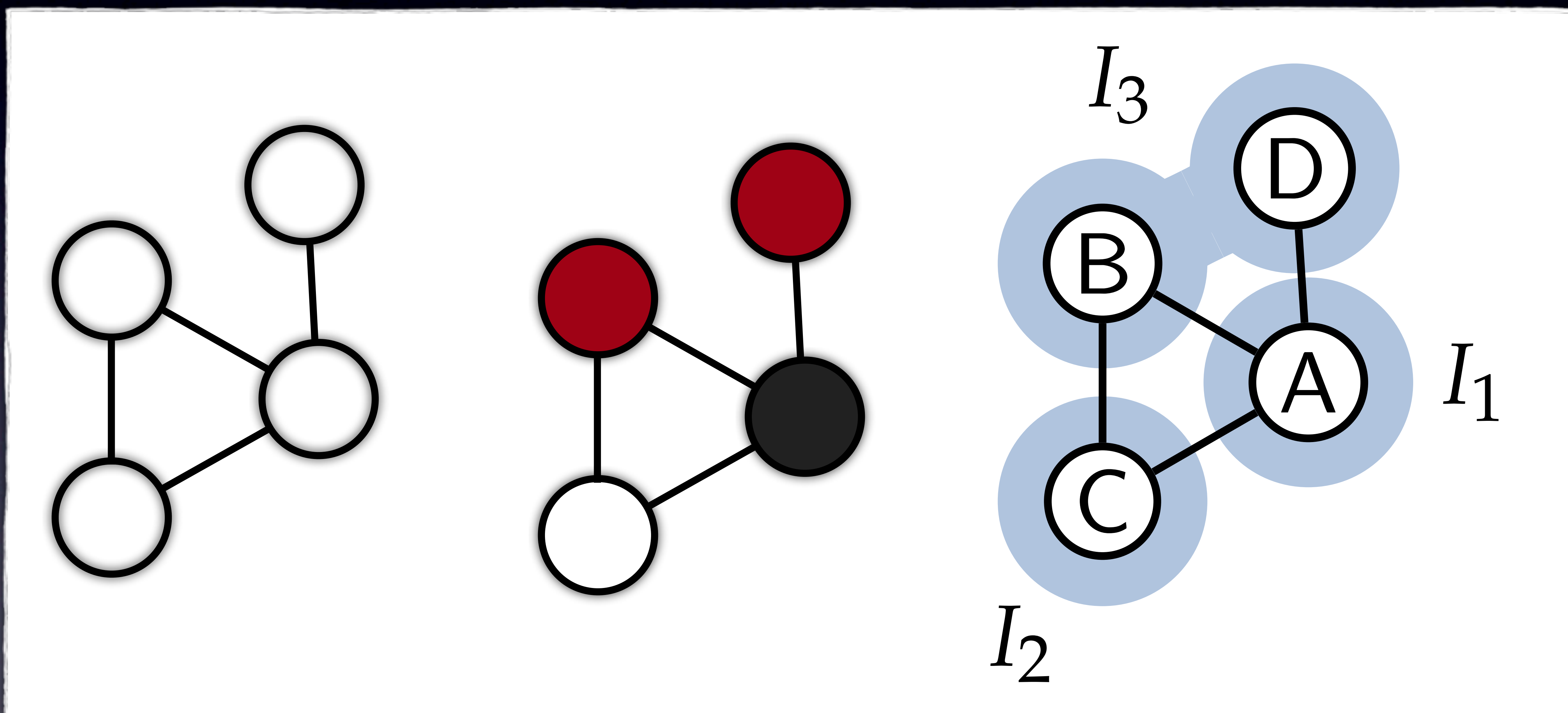


Ryser



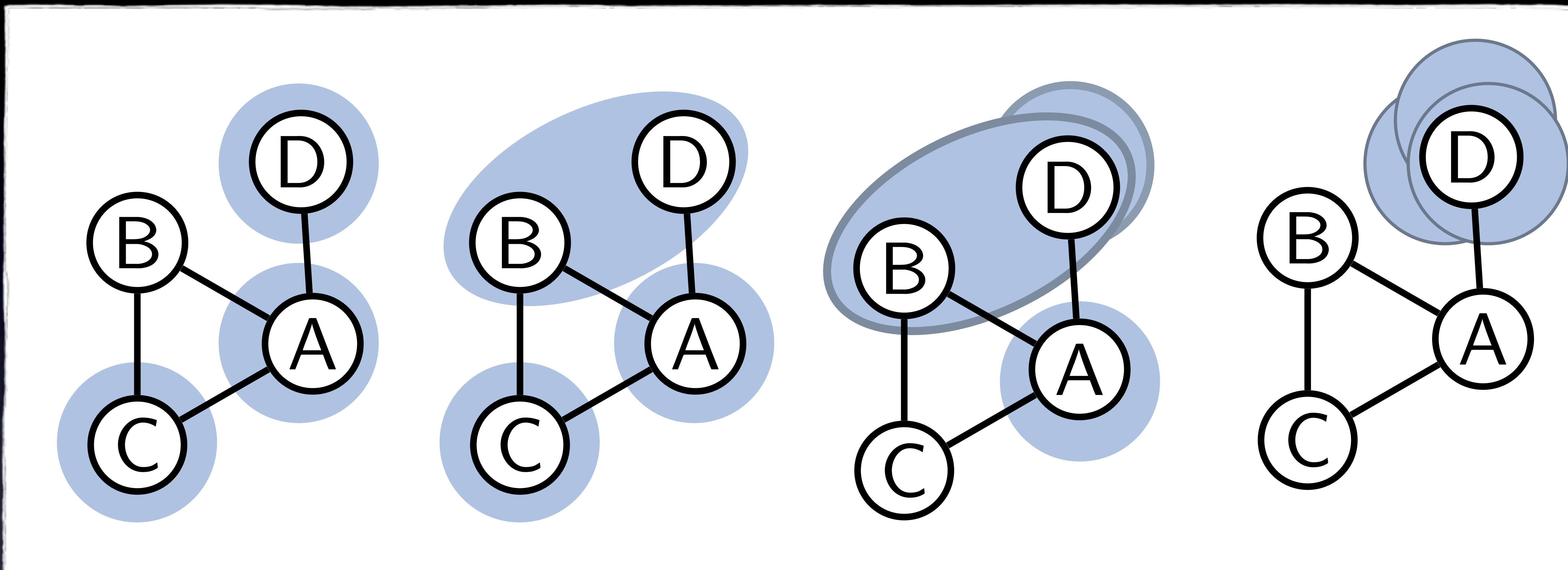


Vertex colouring



I_3
D
B
A
I_1
C
I_2

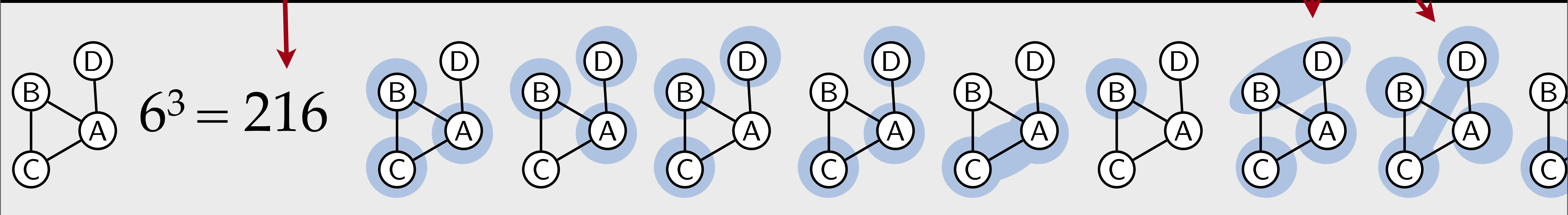
Picking 3 independent sets



A	B	C	D	AB	AC	AD	BC	BD	CD	ABC	ABD	ACD	BCD	ABCD
1	1	1	1	0	0	0	0	1	1	0	0	0	0	0

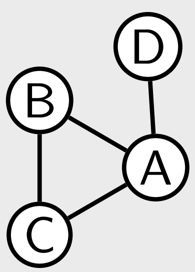
ways pick 3 indep. sets

actually useful ones

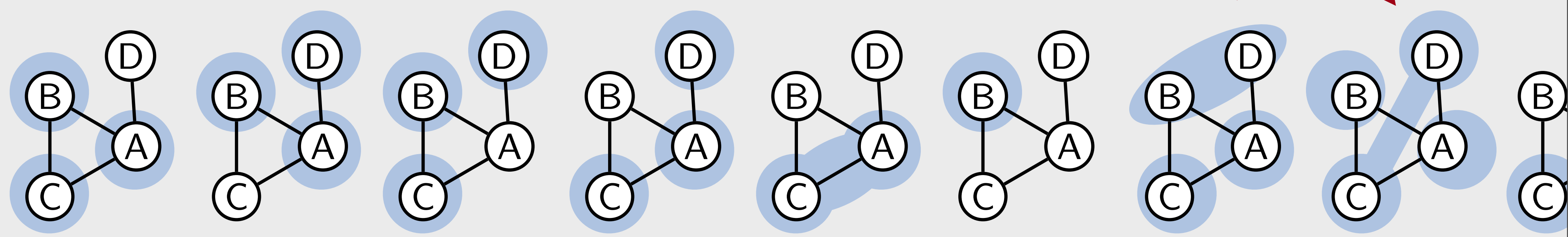


$\#$ ways pick 3 indep. sets

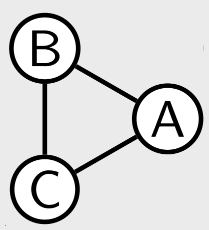
actually useful ones



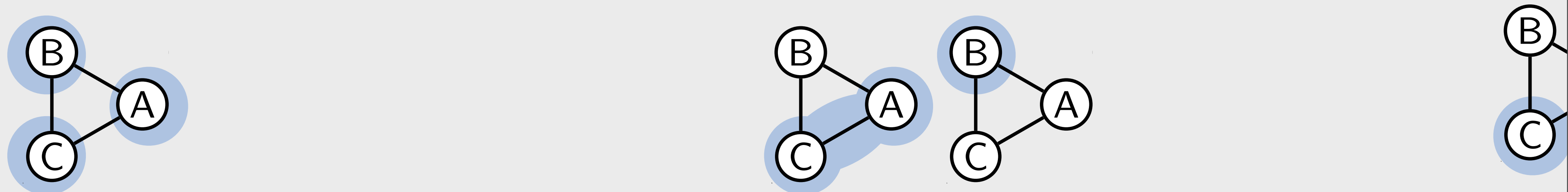
$6^3 = 216$



vertex subsets

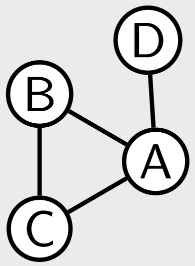


$3^3 = 27$

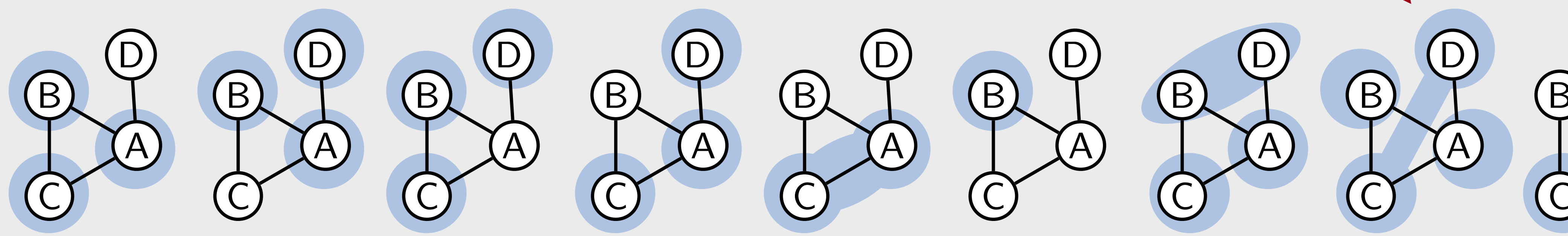


ways pick 3 indep. sets

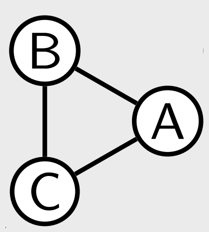
actually useful ones



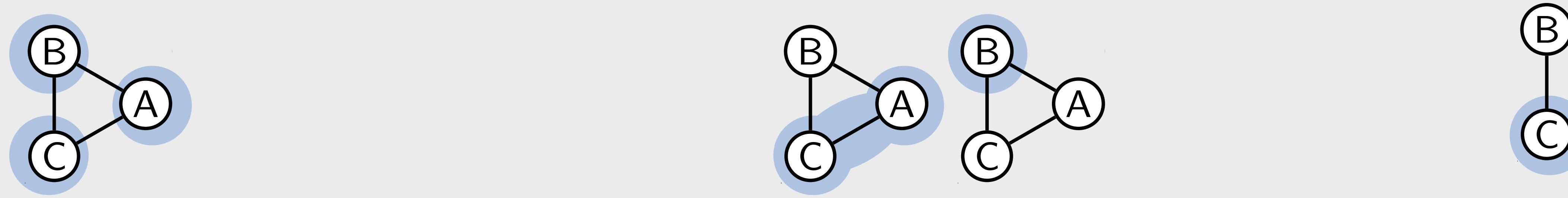
$6^3 = 216$

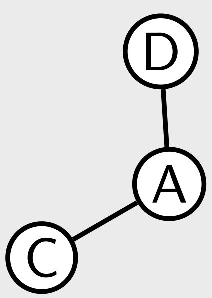


vertex subsets



$3^3 = 27$

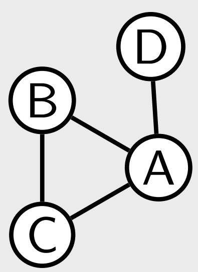




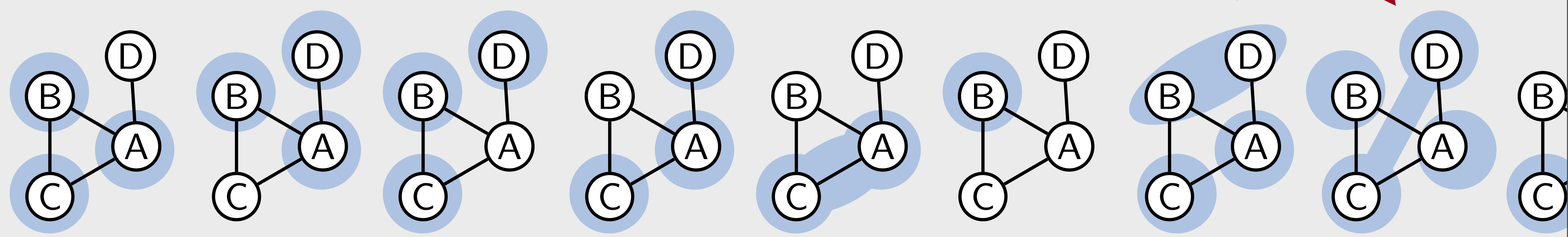
$4^3 = 64$

 # ways pick 3 indep. sets

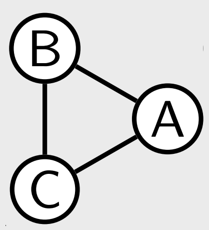
actually useful ones



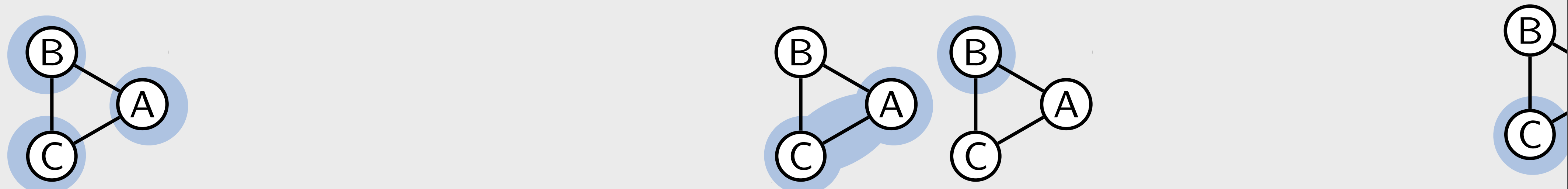
$6^3 = 216$

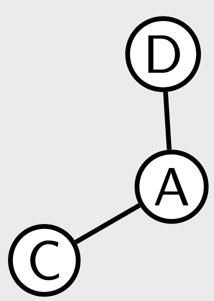


vertex subsets

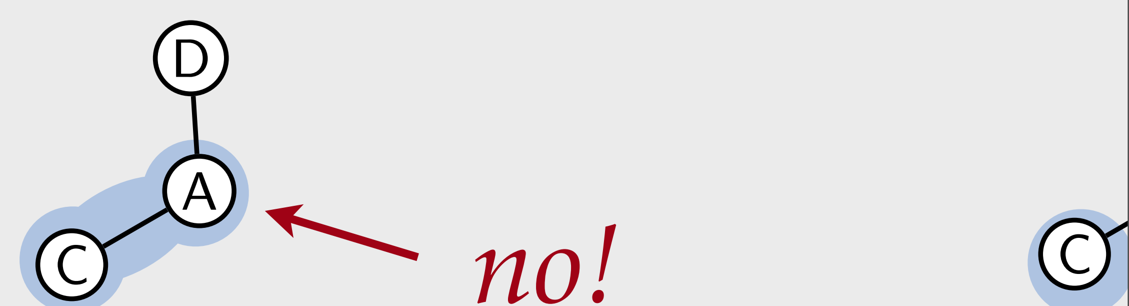


$3^3 = 27$

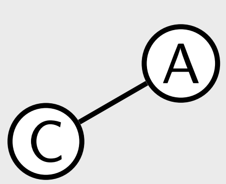




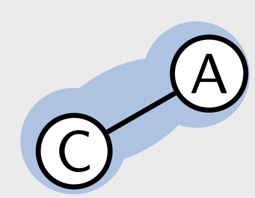
$4^3 = 64$



no!



$2^3 = 8$



yes!

Vertex subset S	# indep. subsets, $g(S)$	$(g(S))^3$
A	1	1
B	1	1
C	1	1
D	1	1
AB	2	8
AC	2	8
AD	2	8
BC	2	8
BD	3	27
CD	3	27
ABC	3	27
ABD	4	64
ACD	4	64
BCD	5	125
ABCD	6	216

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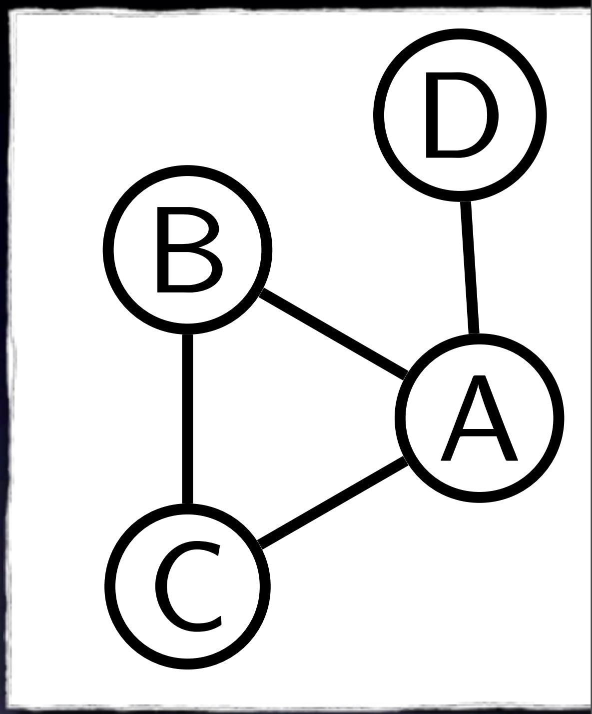

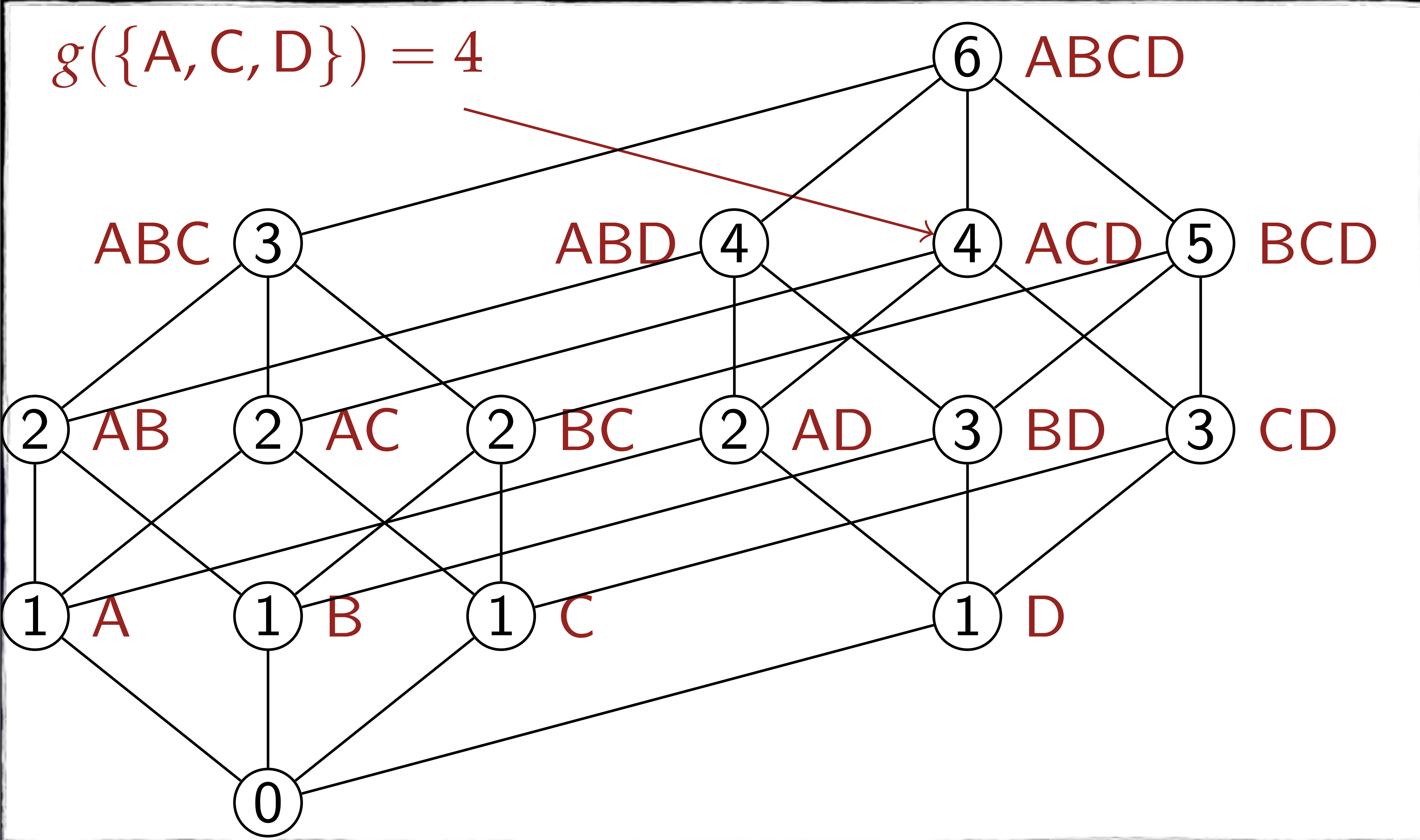

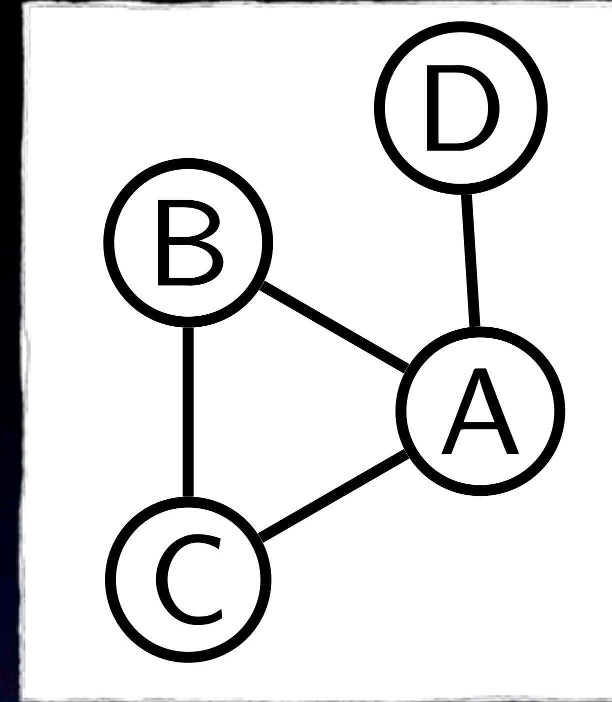

Vertex subset S	# indep.subsets, $g(S)$	$(g(S))^3$
A	1	1
B	1	1
C	1	1
D	1	1
AB	2	8
AC	2	8
AD	2	8
BC	2	8
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CD	3	27
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ABD	4	64
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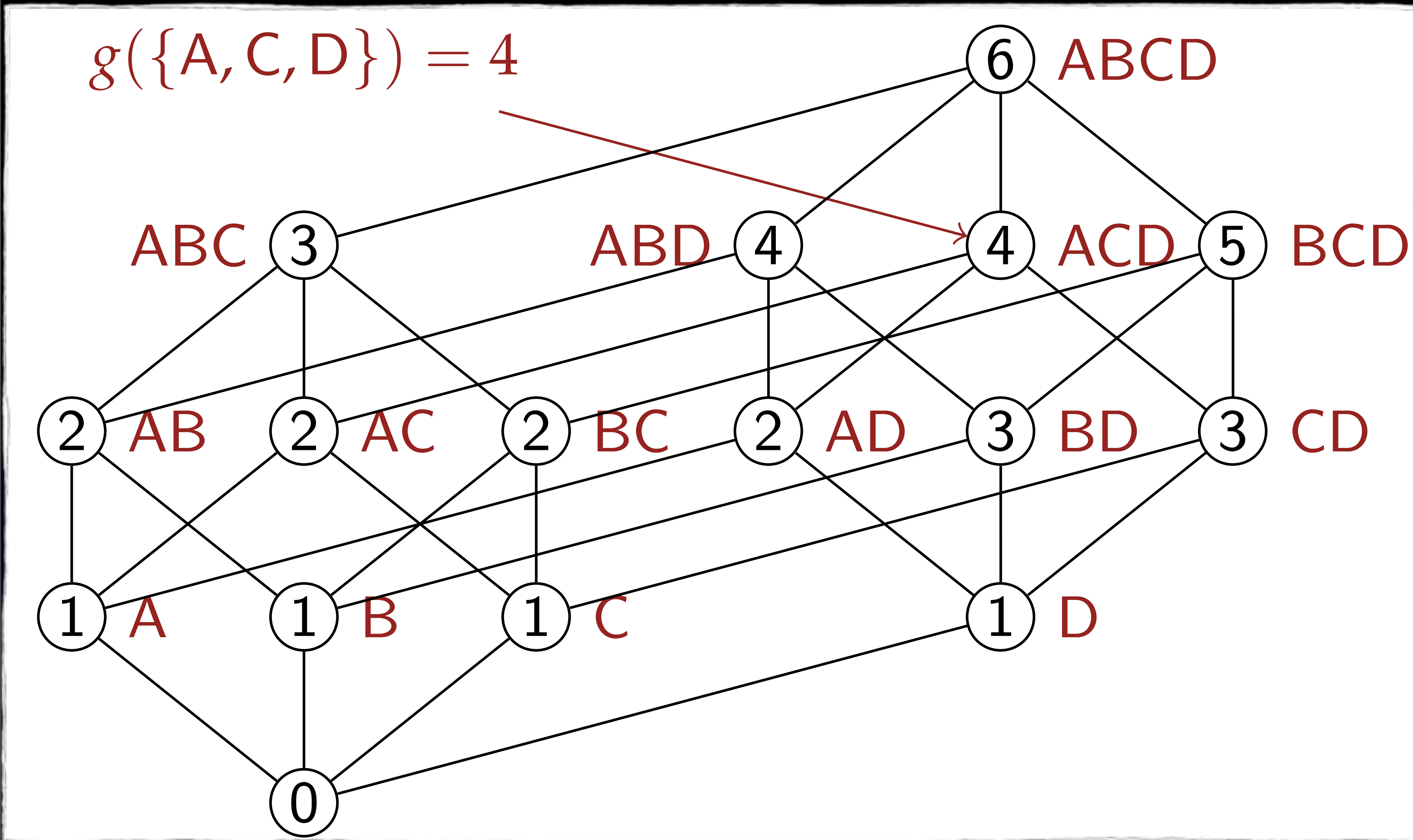
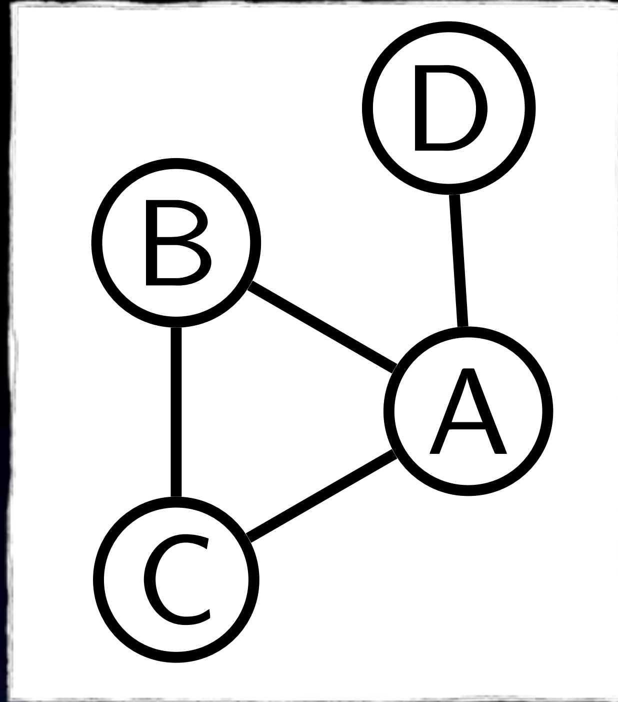
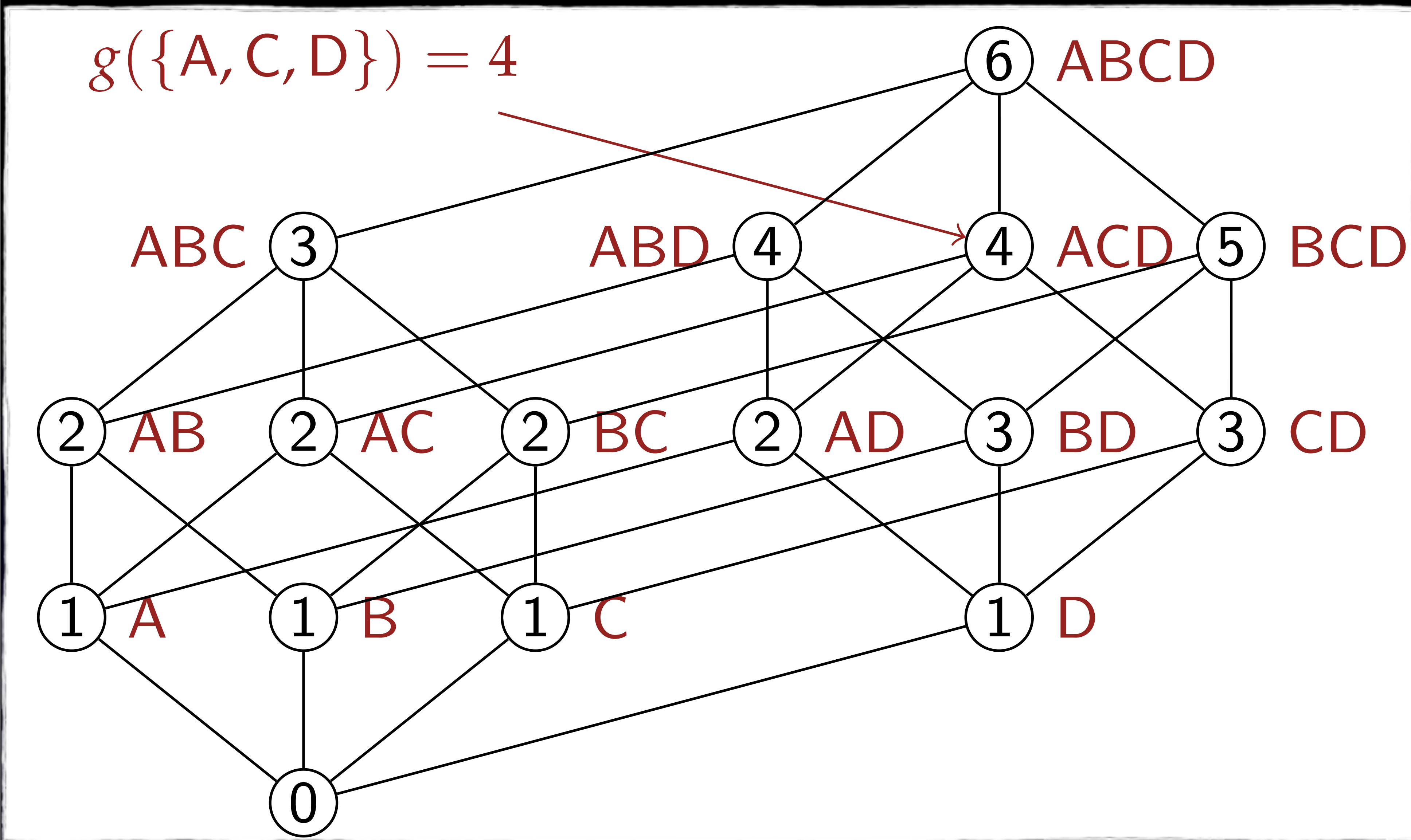
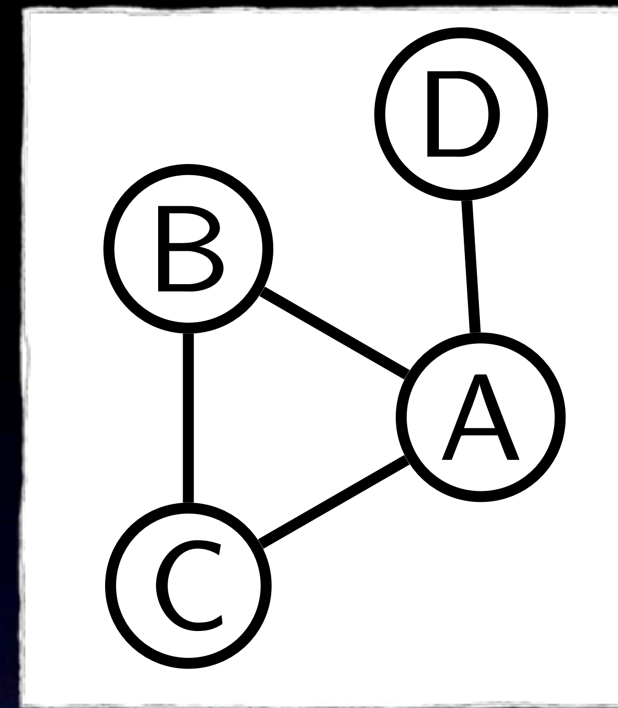
ABCD	6	216


$g(\{A, C, D\}) = 4$
6 ABCD
ABC 3
ABD 4
4 ACD
5 BCD
2 AB
2 AC
2 BC
2 AD
3 BD
3 CD
1 A
1 B
1 C
1 D
0

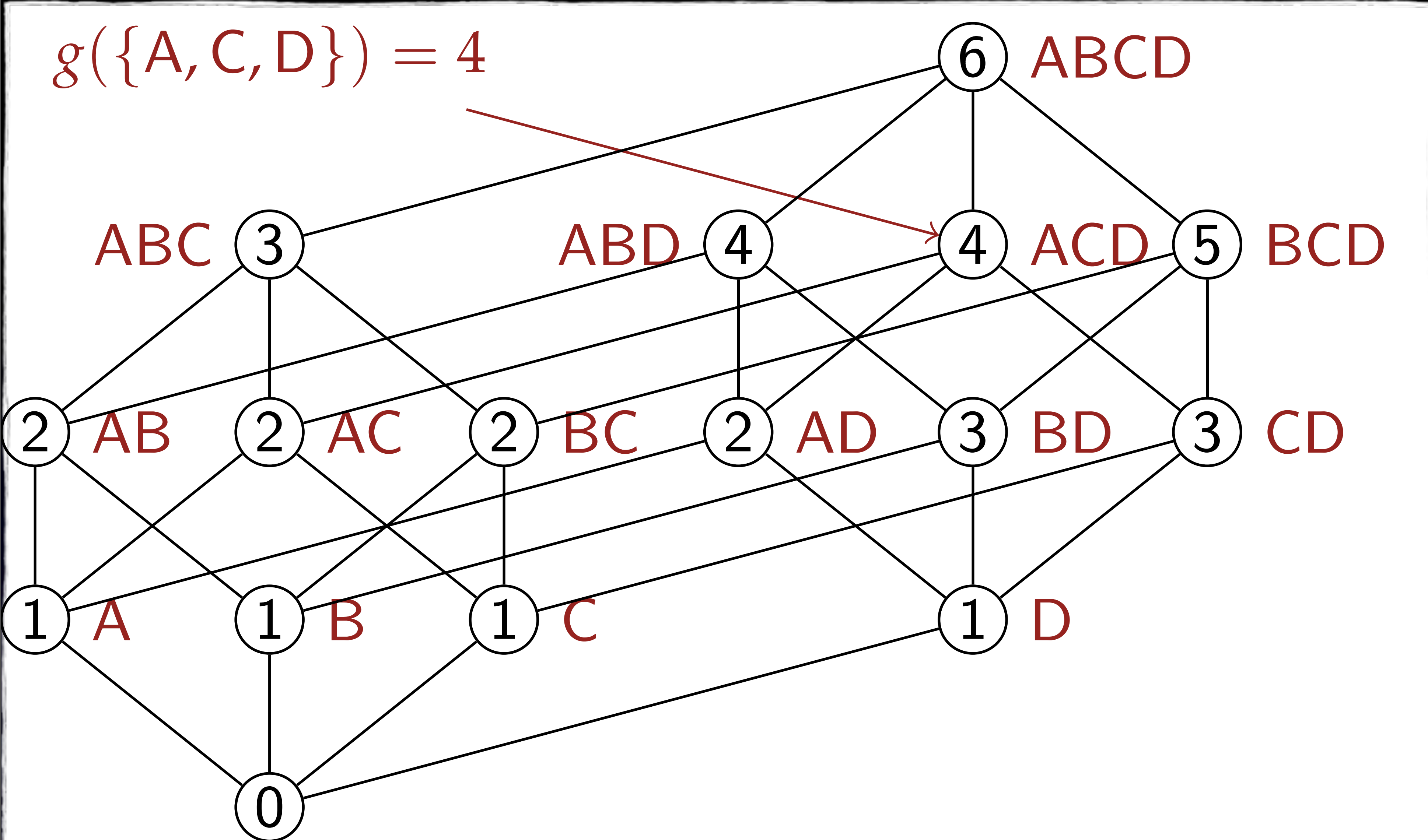

D
B
A
C

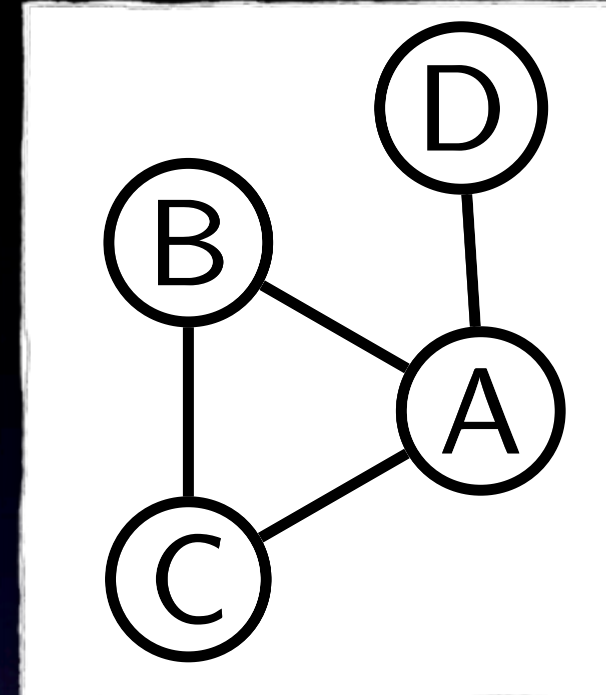




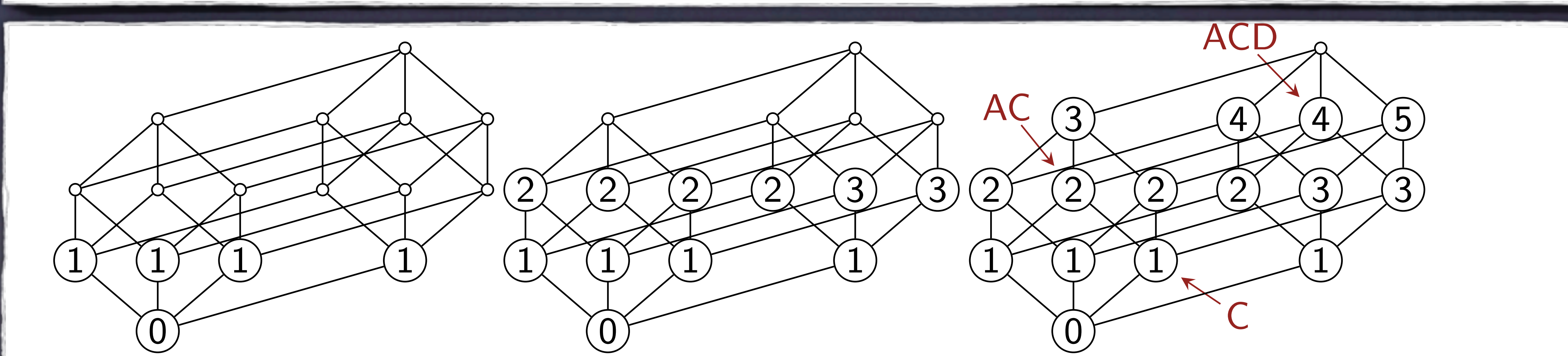
$$\sum_{S \subseteq N} 2^{|S|} = \sum_{i=1}^{n} \binom{n}{i} 2^i = 3^n$$

polynomial space





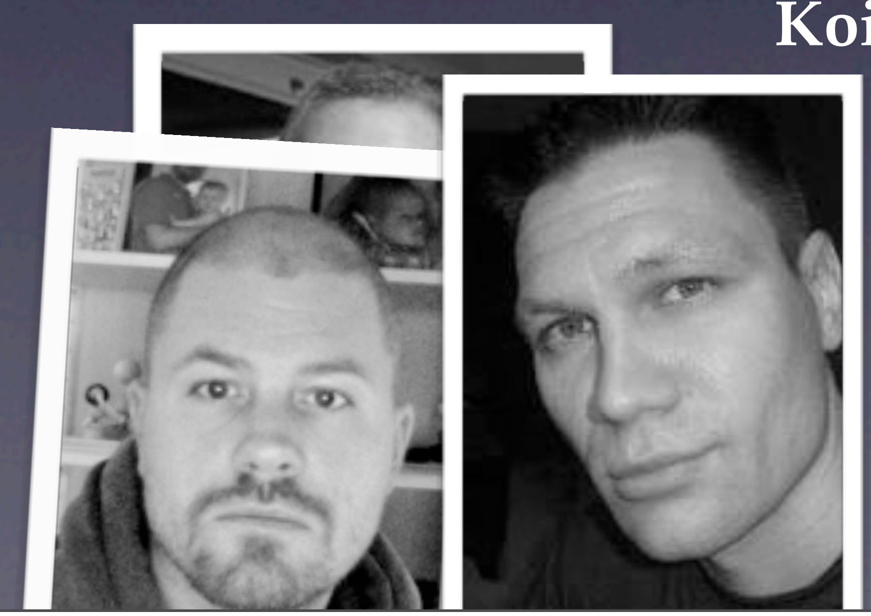
$$g(S) = g(S \setminus \{v\}) + g(S \setminus N[v]) + 1 \qquad (v \in S)$$



Graph colouring

Compute $\sum_{S \subseteq N} (-1)^{n-|S|} (g(S))^k$

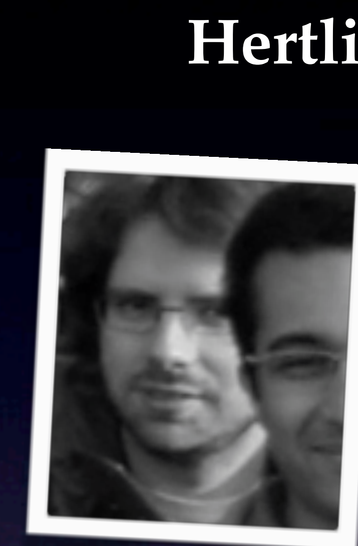
$O^*(3^n)$ time polynomial space	$O^*(2^n)$ time $O^*(2^n)$ space



Exponential Time Hypothesis

Exponential Time Hypothesis

Can do 3-Sat *in*
time 1.308^n



Exponential Time Hypothesis

Hertli

Can do 3-Sat *in time* 1.308^n

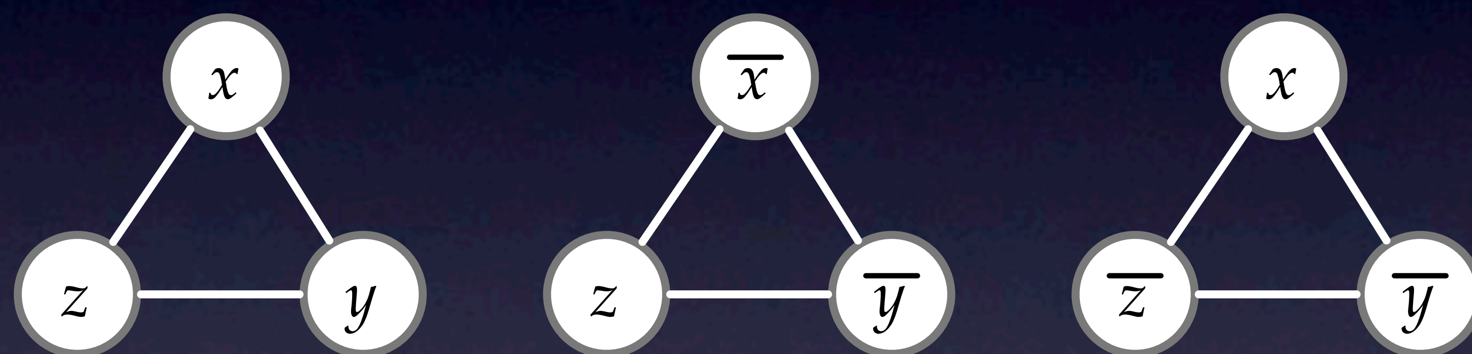
Zane

Impagliazzo

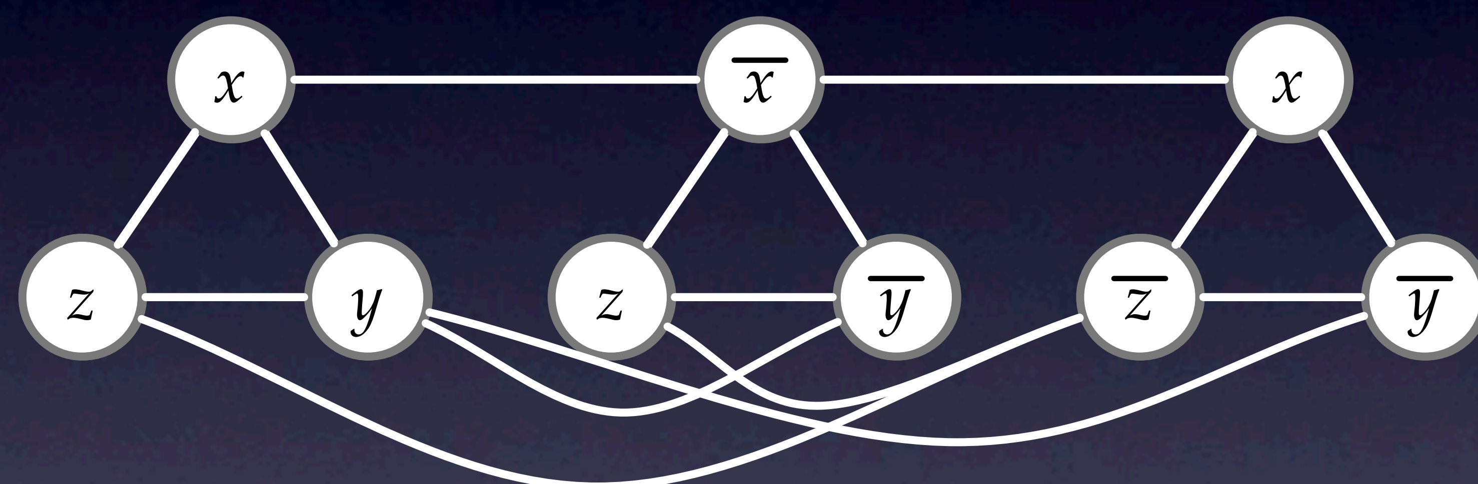
Paturi

Can't do 3-Sat *in time* $\exp(o(n))$

$$(x \vee y \vee z) \wedge (\overline{x} \vee \overline{y} \vee z) \wedge (x \vee \overline{y} \vee \overline{z})$$



$$(x \vee y \vee z) \wedge (\overline{x} \vee \overline{y} \vee z) \wedge (x \vee \overline{y} \vee \overline{z})$$



n vars

m clauses

$3m=O(n^3)$ verts

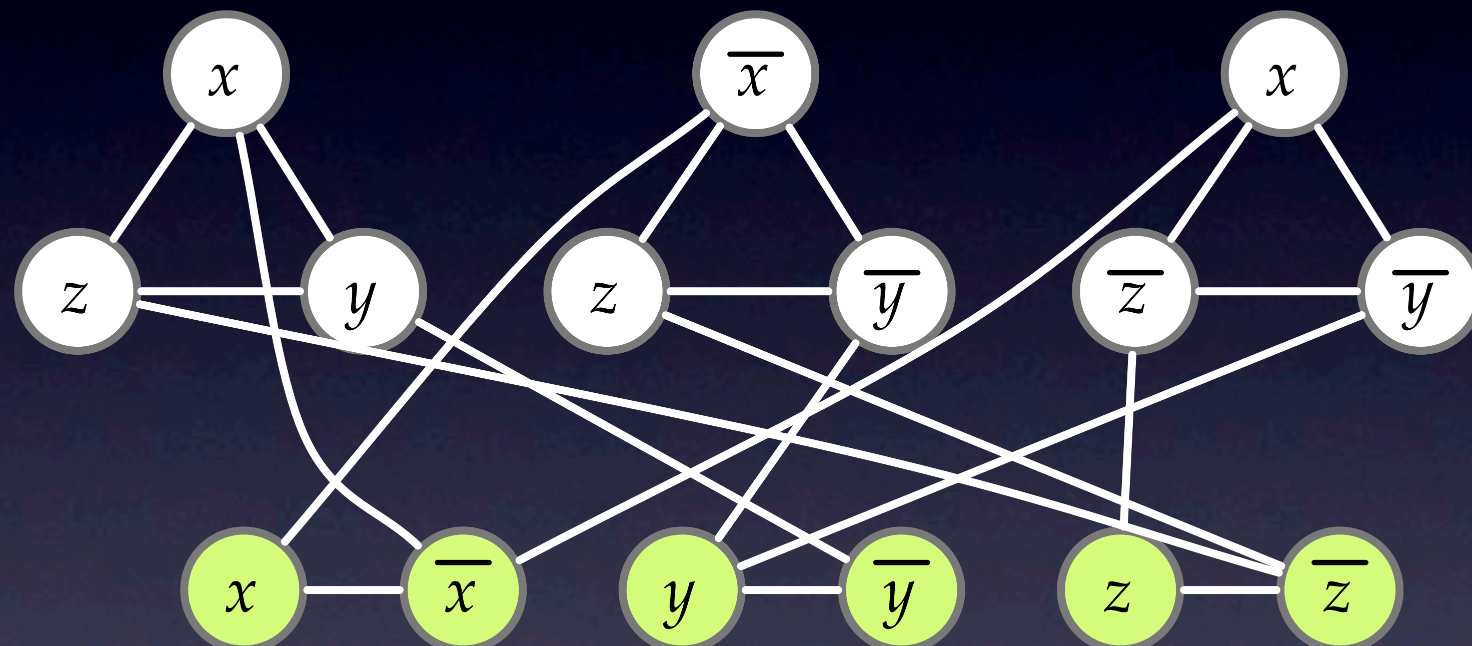
$O(m^2)$ edges

$\exp(o(n))$ alg for 3-SAT

$\exp(o(n^{1/3}))$ alg for I.S.

$\exp(o(m^{1/2}))$ alg for I.S.

$$(x \vee y \vee z) \wedge (\overline{x} \vee \overline{y} \vee z) \wedge (x \vee \overline{y} \vee \overline{z})$$



n vars

m clauses

exp(o(n)) alg for 3-SAT

$3m = O(n^3)$ verts

O(m) edges

exp(o($n^{1/3}$)) alg for I.S.

exp(o(m)) alg for I.S.

Independent set
n vertices m edges

1.1888^n

c^m

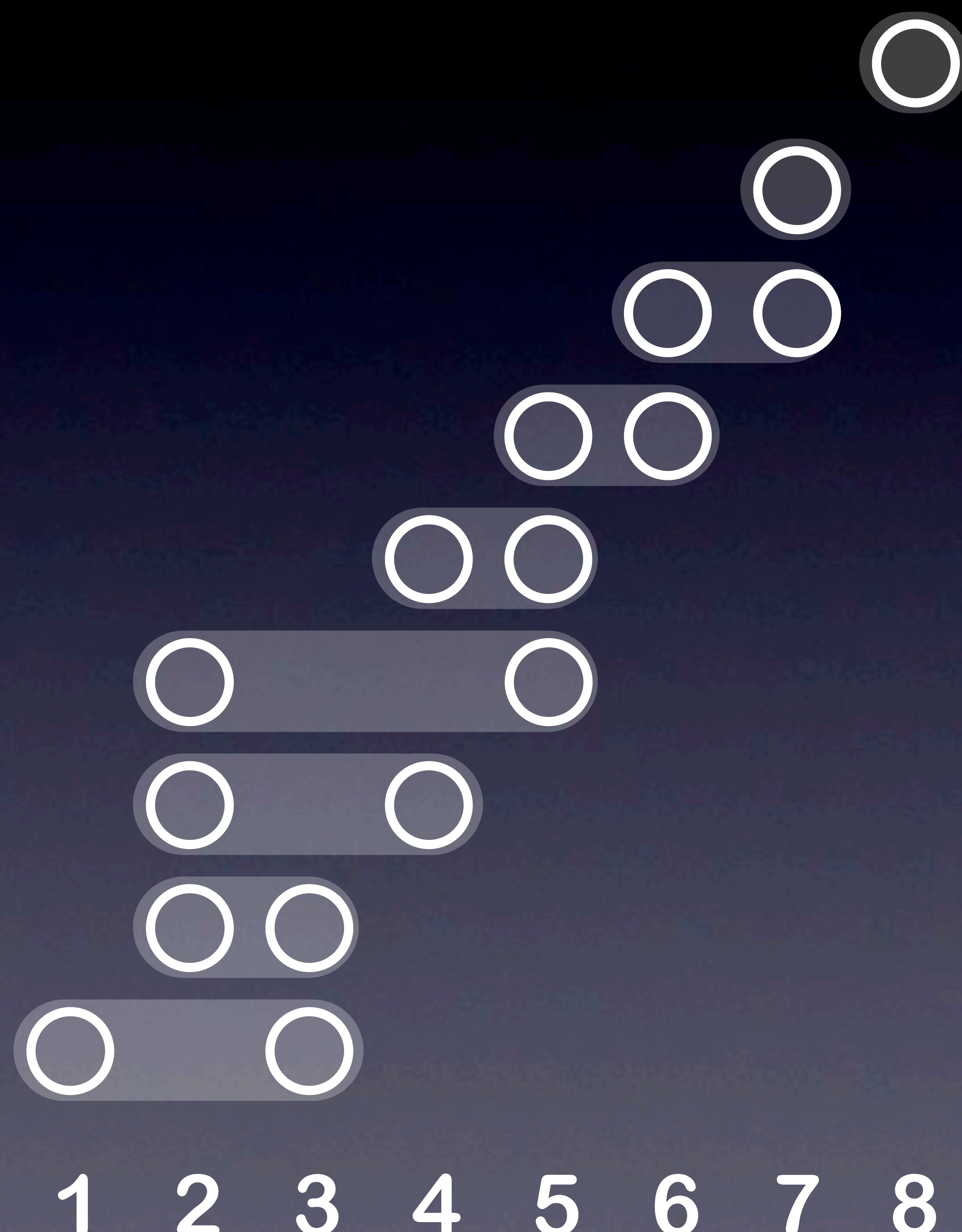
Clique
n vertices m edges

1.1888^n

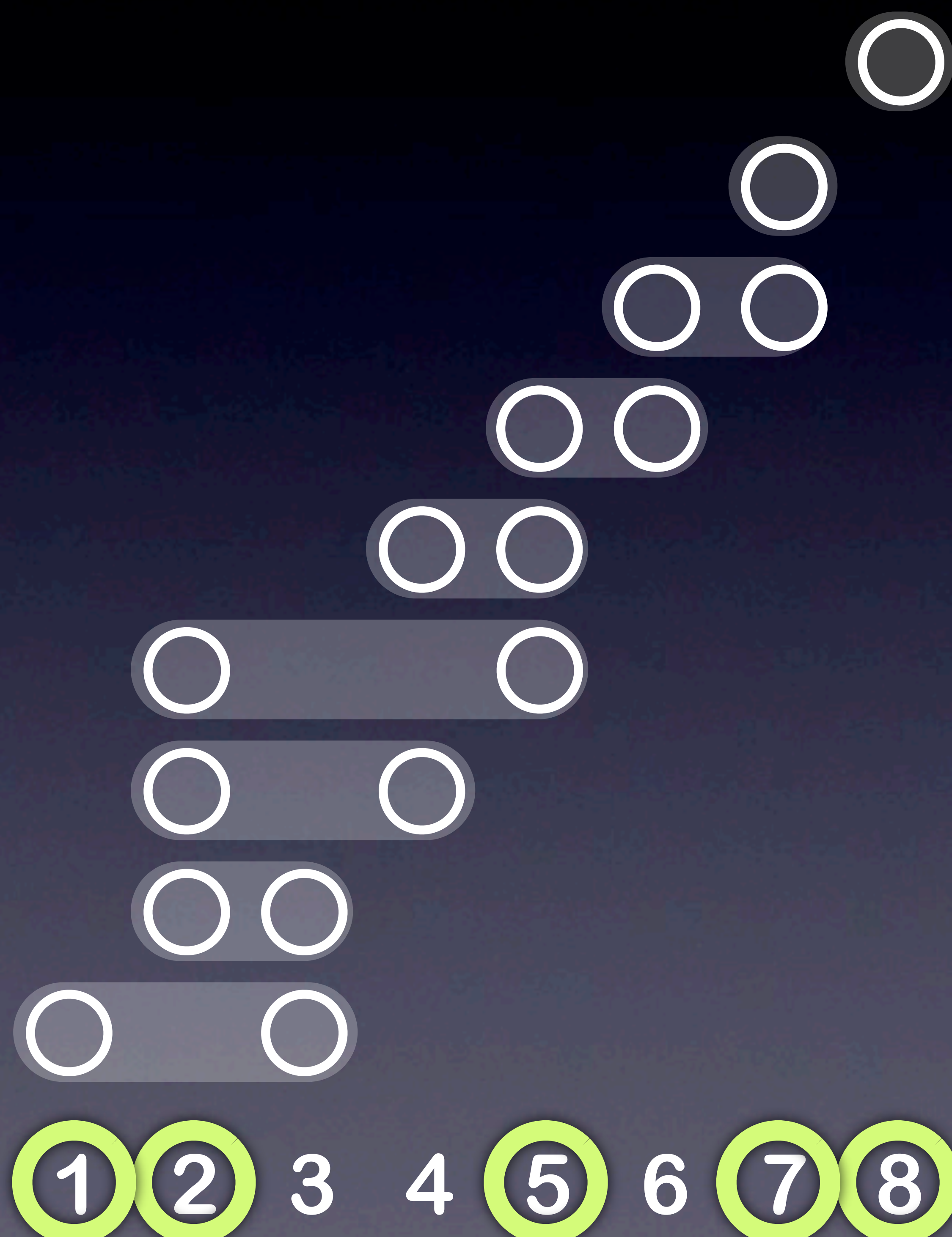
$2^{\sqrt{m}\log n}$

Sparsification

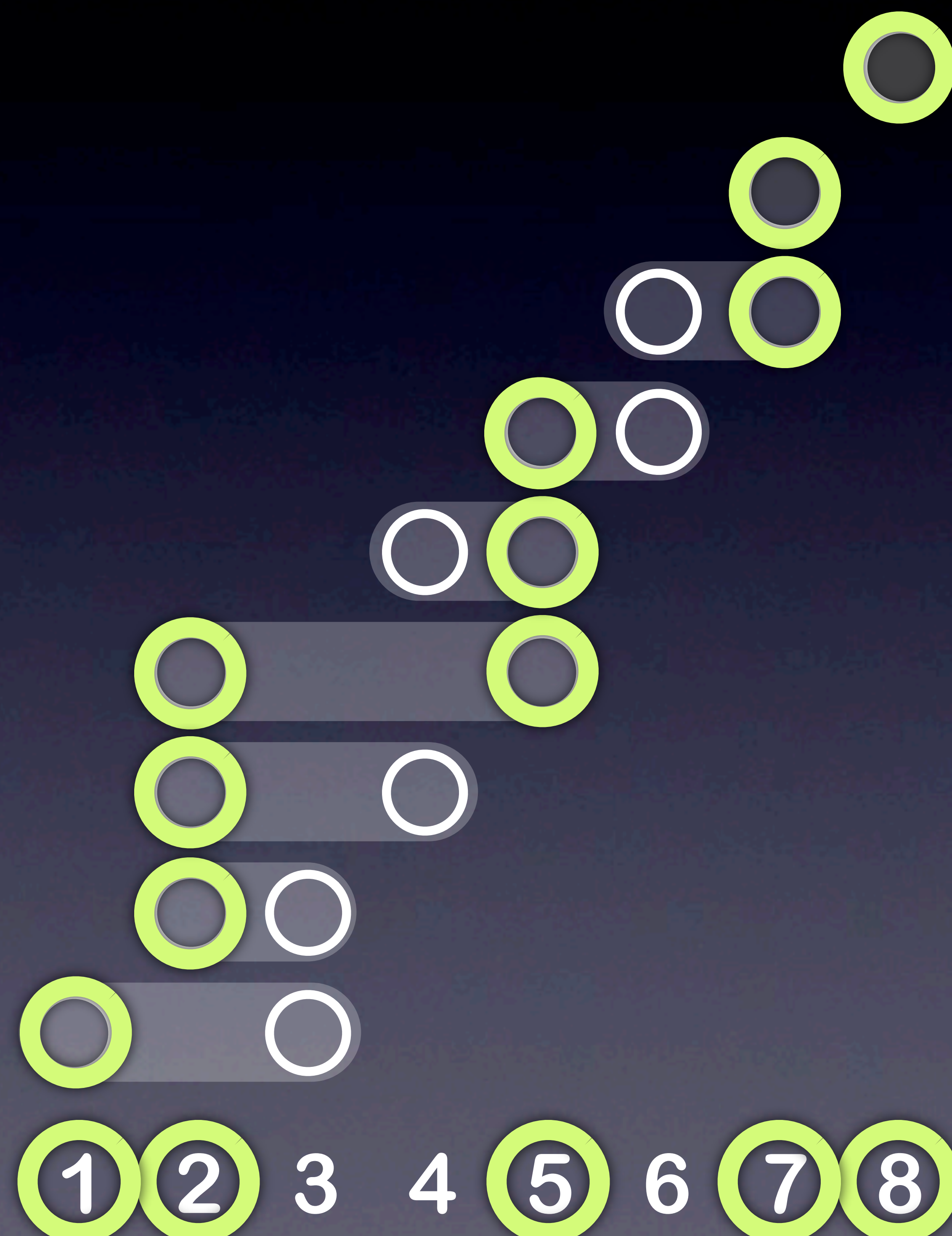
Hitting Set



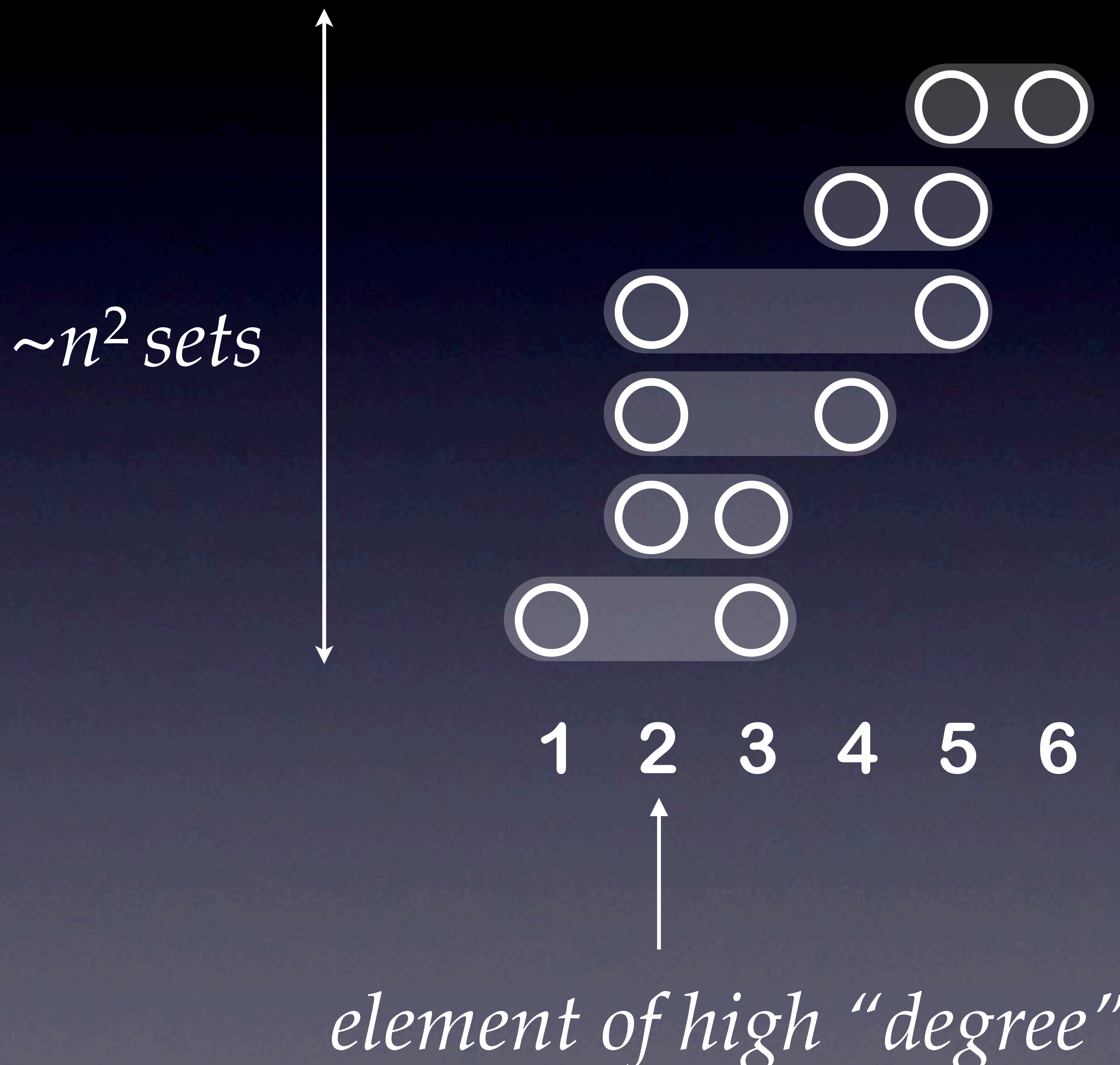
Hitting Set



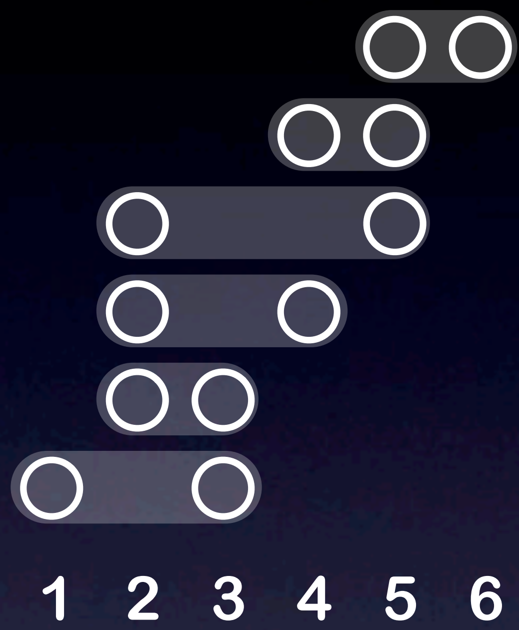
Hitting Set



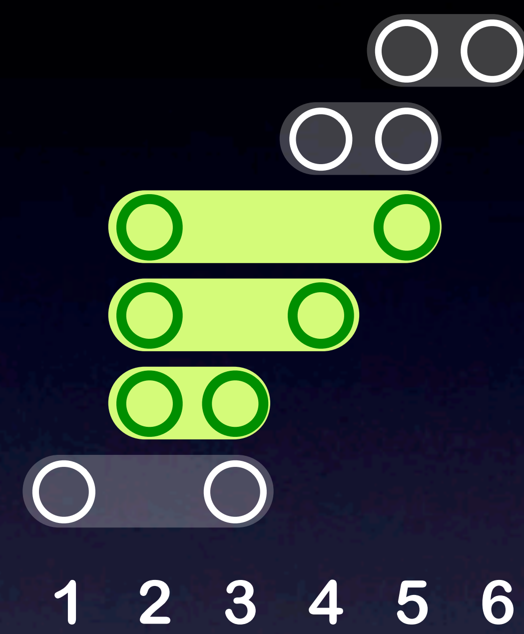
Sparsifying a Hitting Set Instance



Sparsifying a Hitting Set Instance



Sparsifying a Hitting Set Instance



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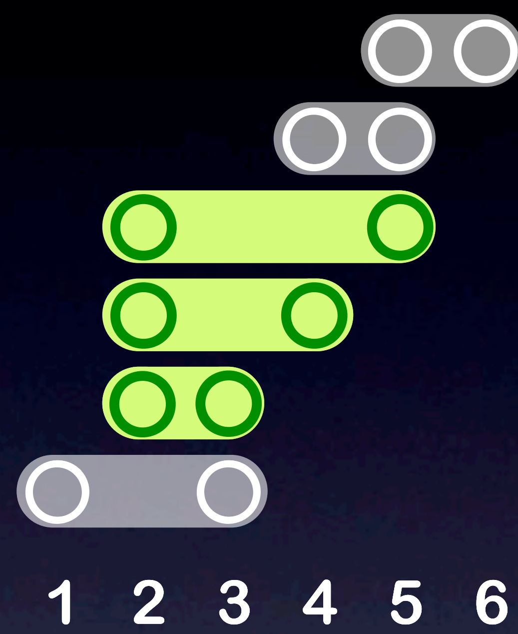

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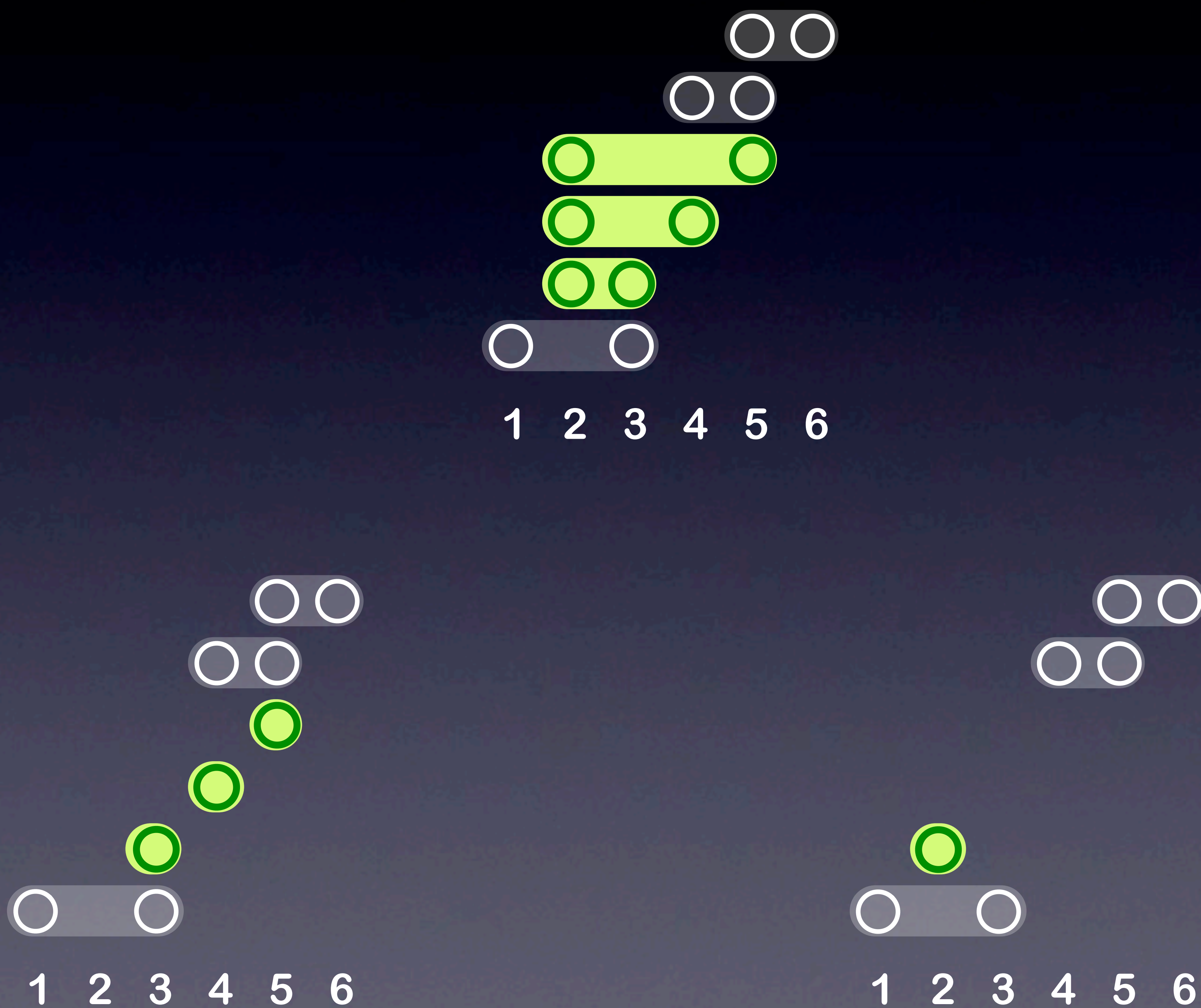
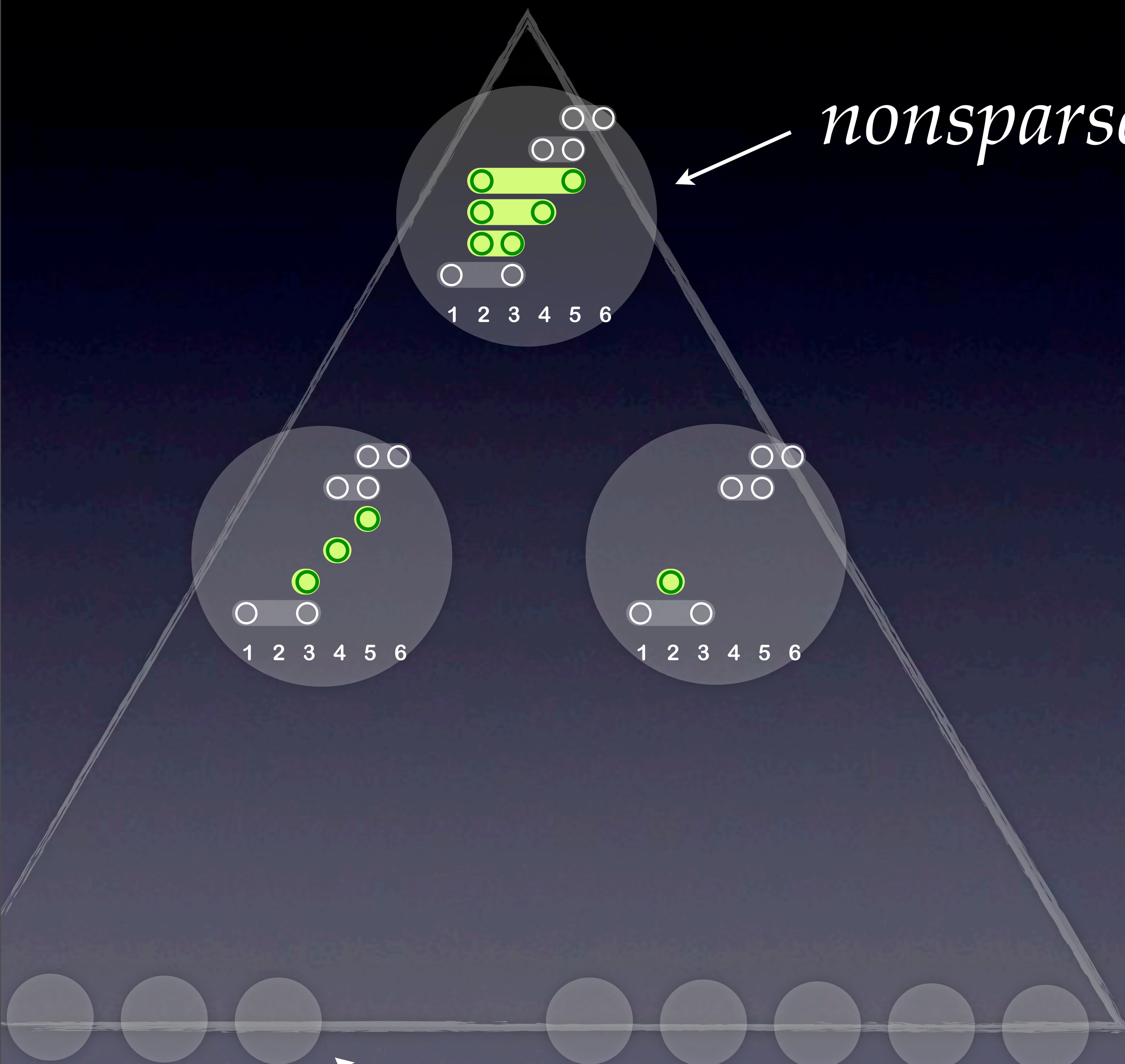


Sparsifying a Hitting Set Instance



nonsparse instance

high degree

max degree r·n

sparse instances

$\exp(o(n)) \cdot 2^n = \exp(n)$

2^n leaves

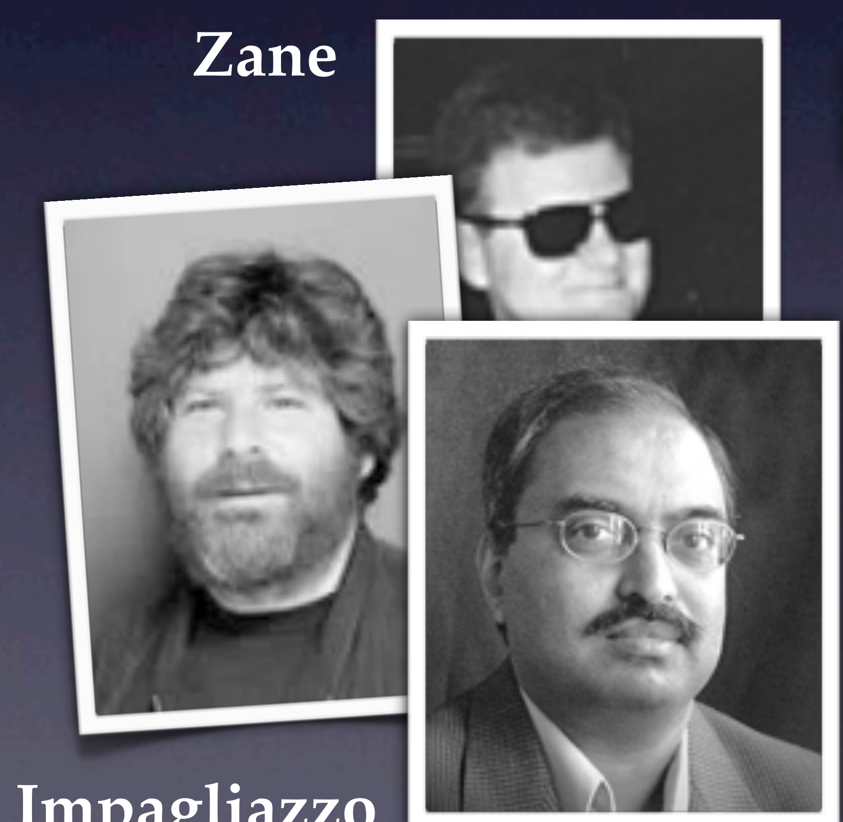
$\exp(o(n)) \cdot \exp(H(1/r)n) =$
$\exp(o(n))$

$C(n,1) + \ldots + C(n,n/r)$
leaves

Exponential Time Hypothesis

Can't do 3-Sat *in time* exp(o(n))

Zane



Can't do 18-Sat *in time* exp(o(m))

Can't do Independent Set *in time* exp(o(n))

Impagliazzo

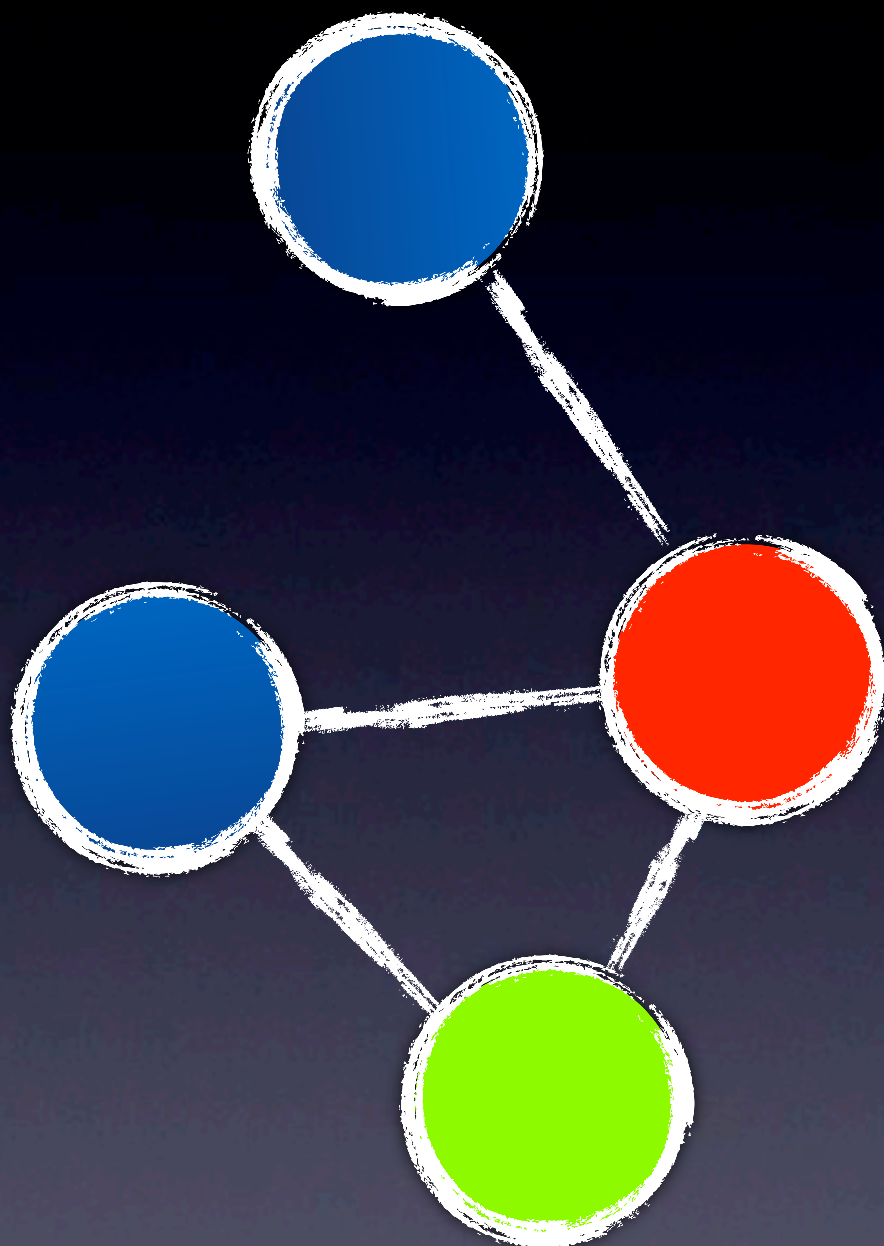
Paturi

Why No Dependency on # Colours is Surprising

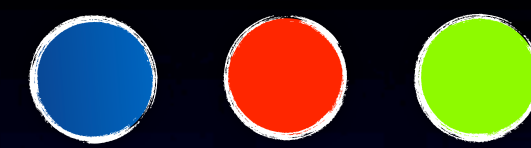
CSP(q,2)

q states, pairwise constraints

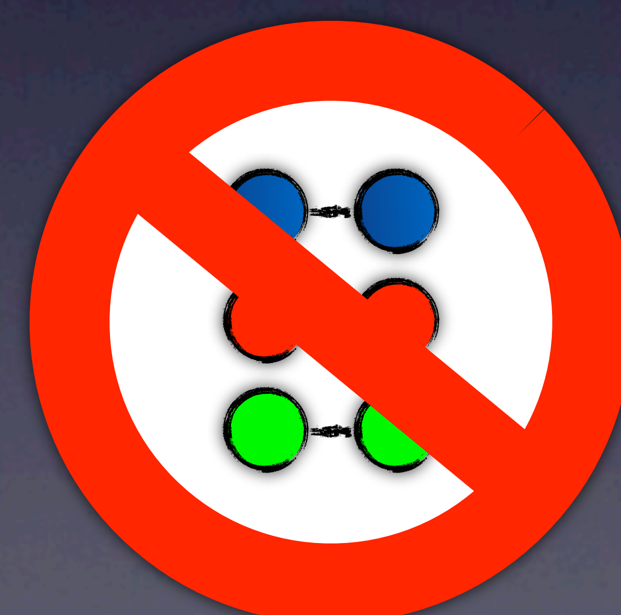
Traxler

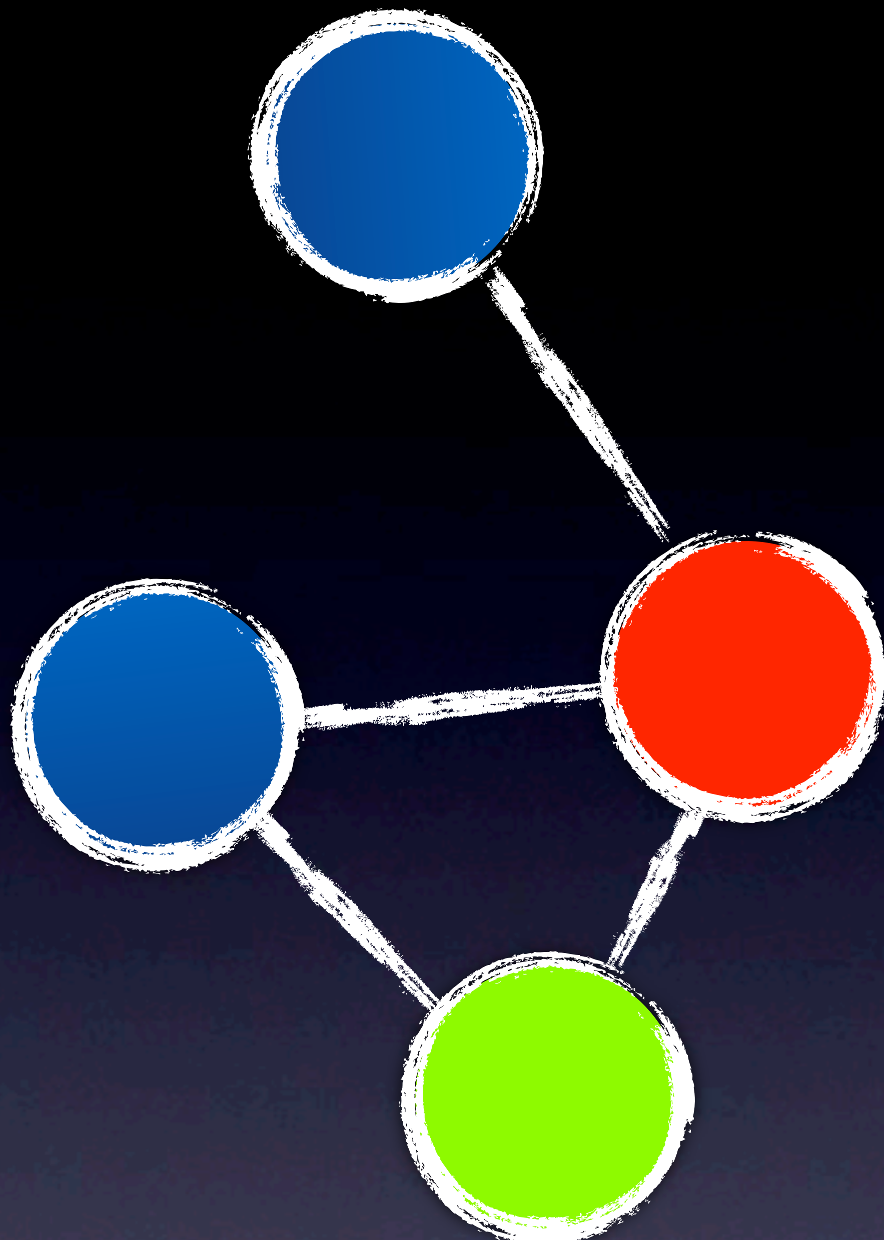


states:

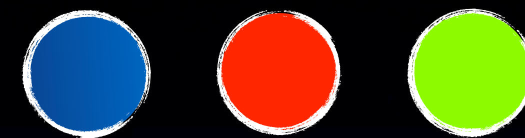


constraints

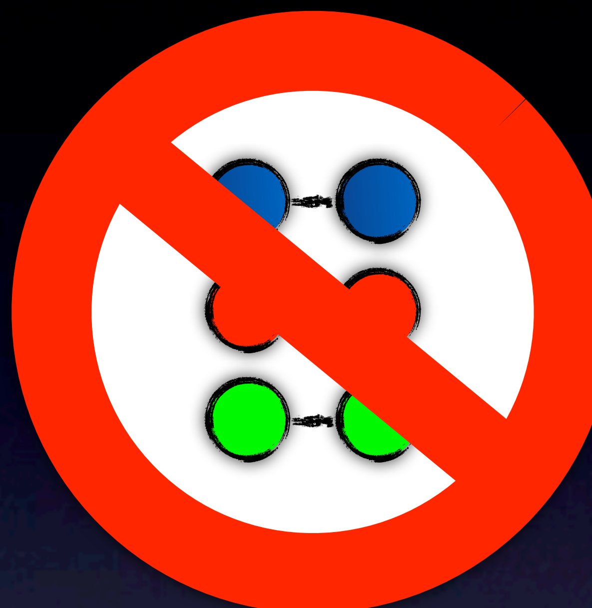


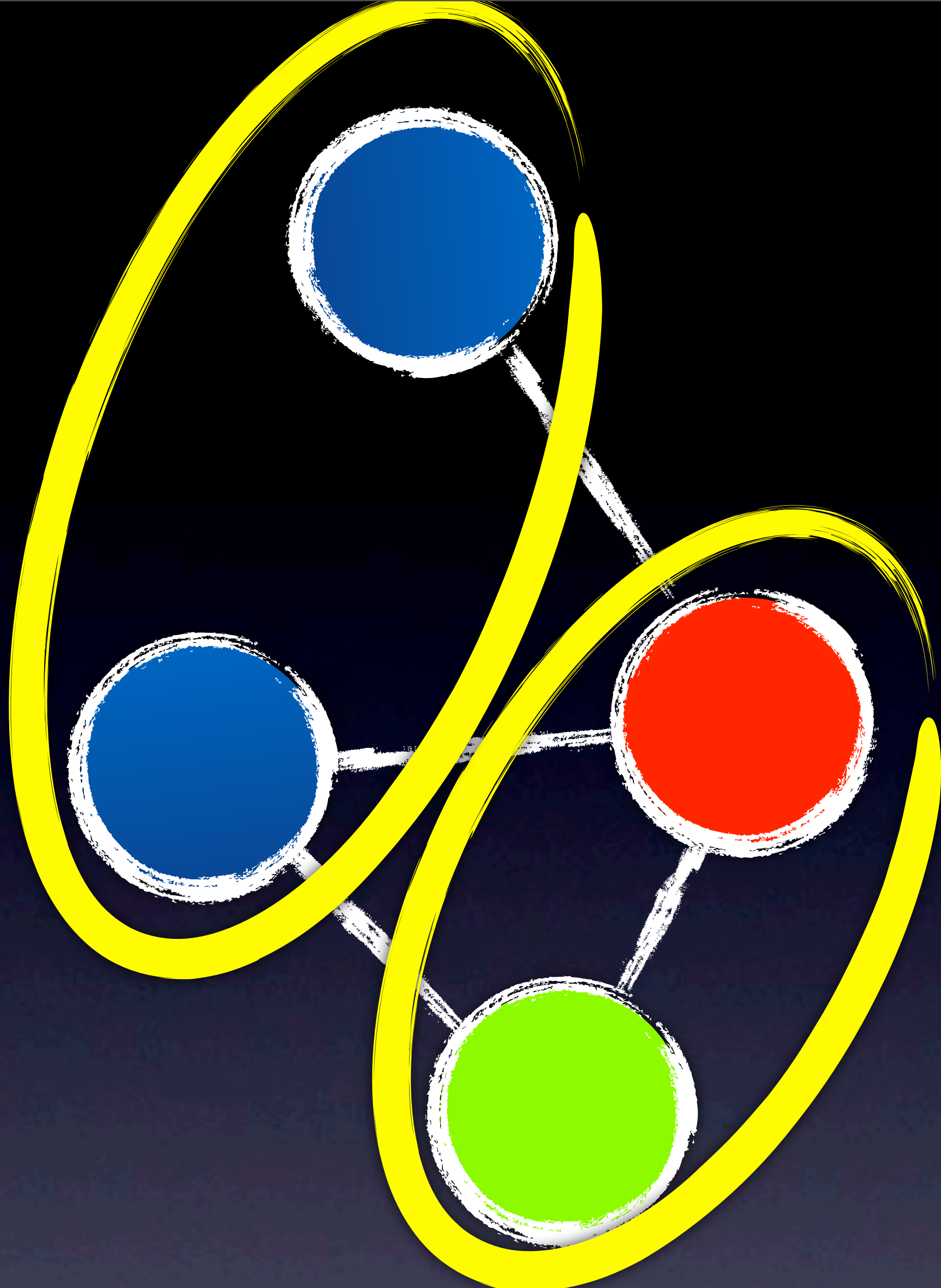


states:

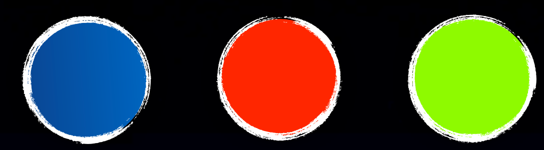


constraints

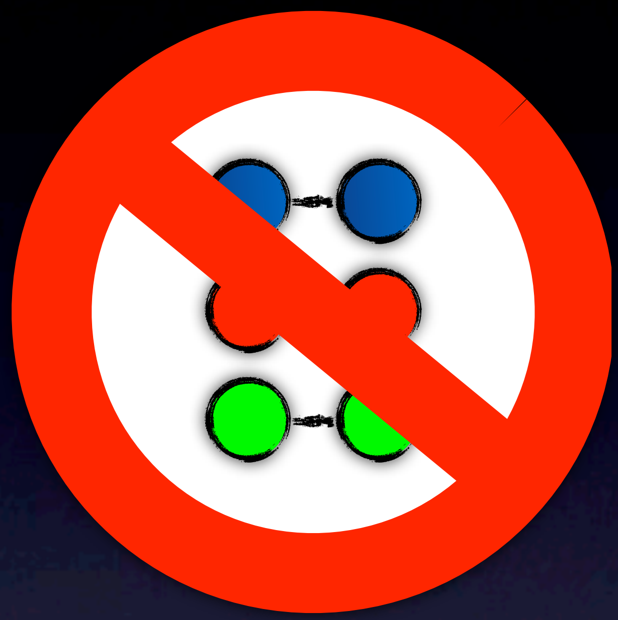


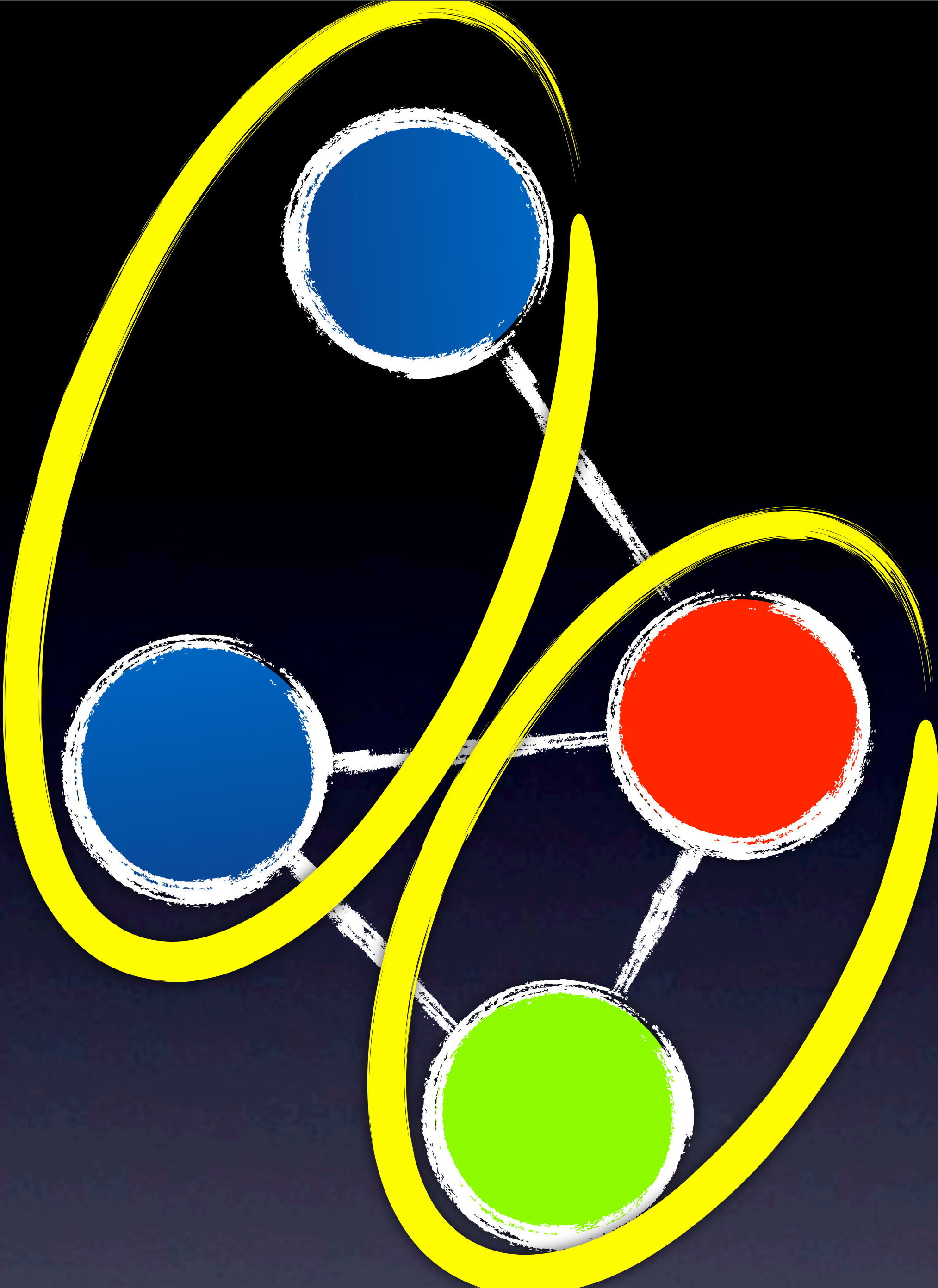


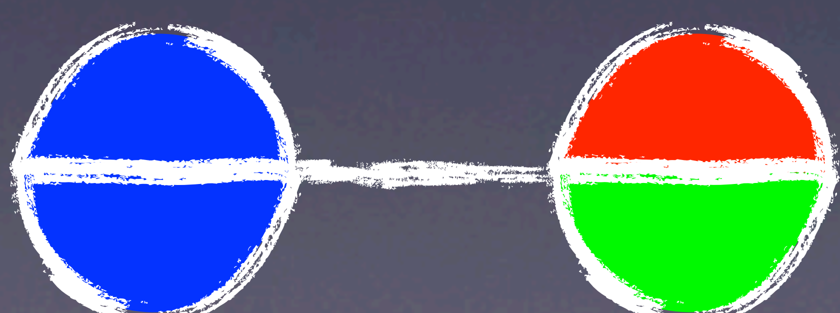
states:



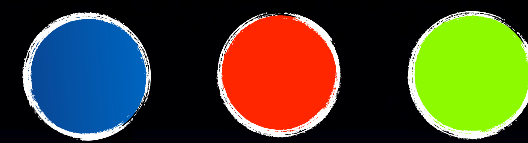
constraints



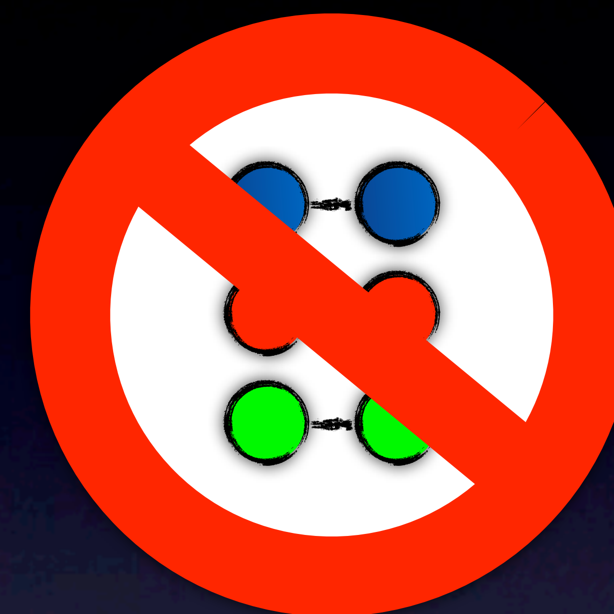


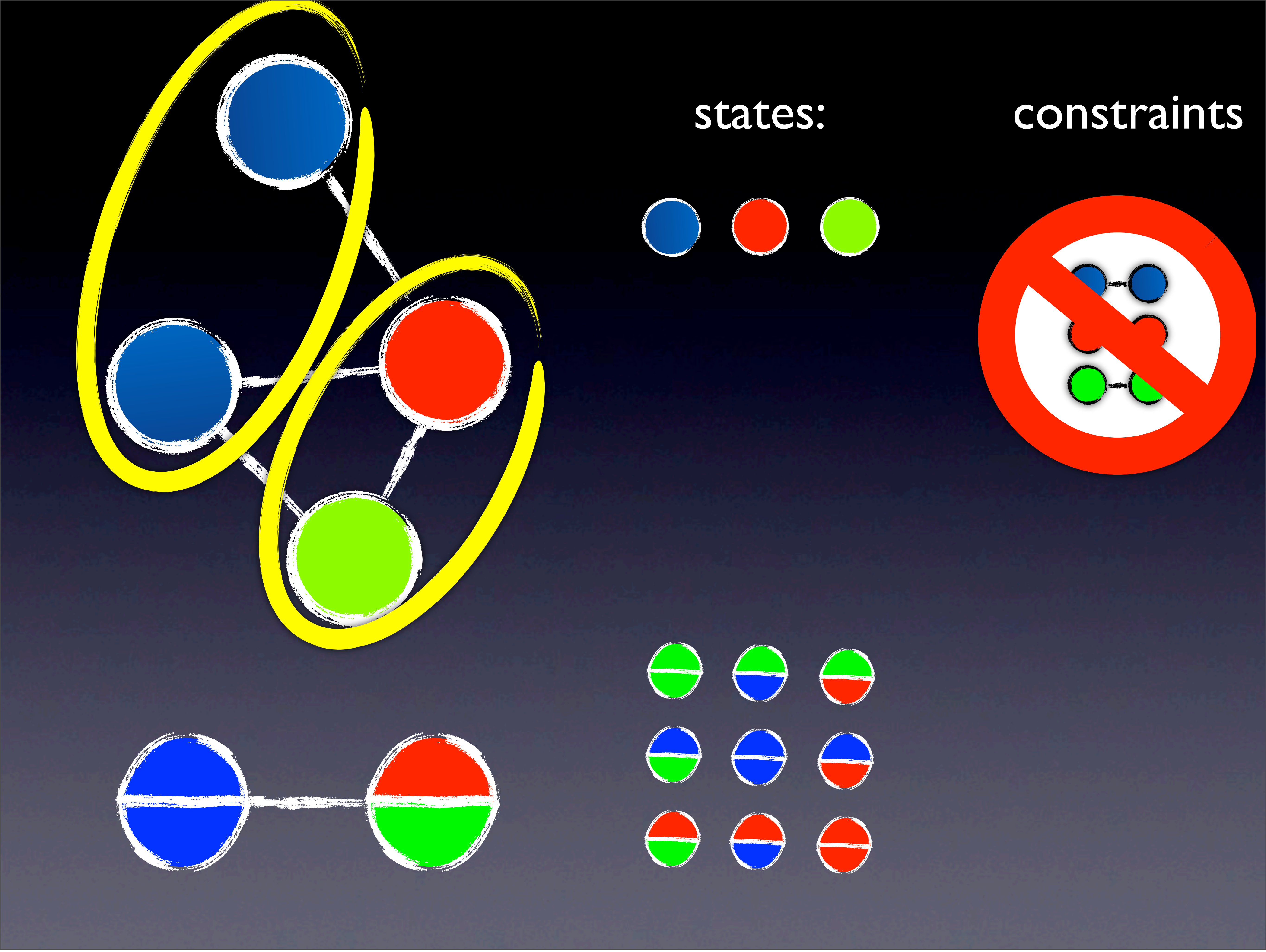


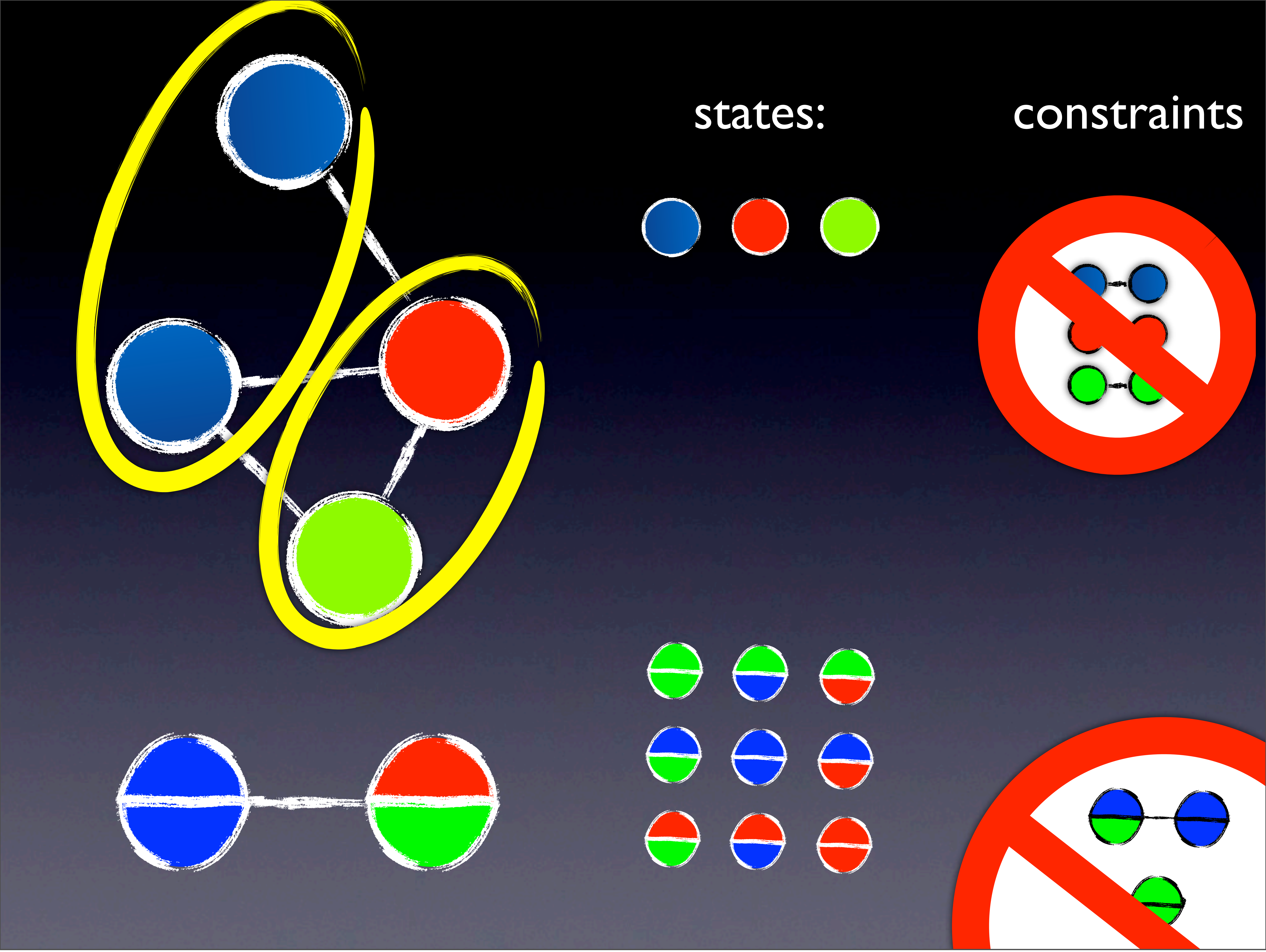
states:



constraints




states:
constraints


states:
constraints

n verts

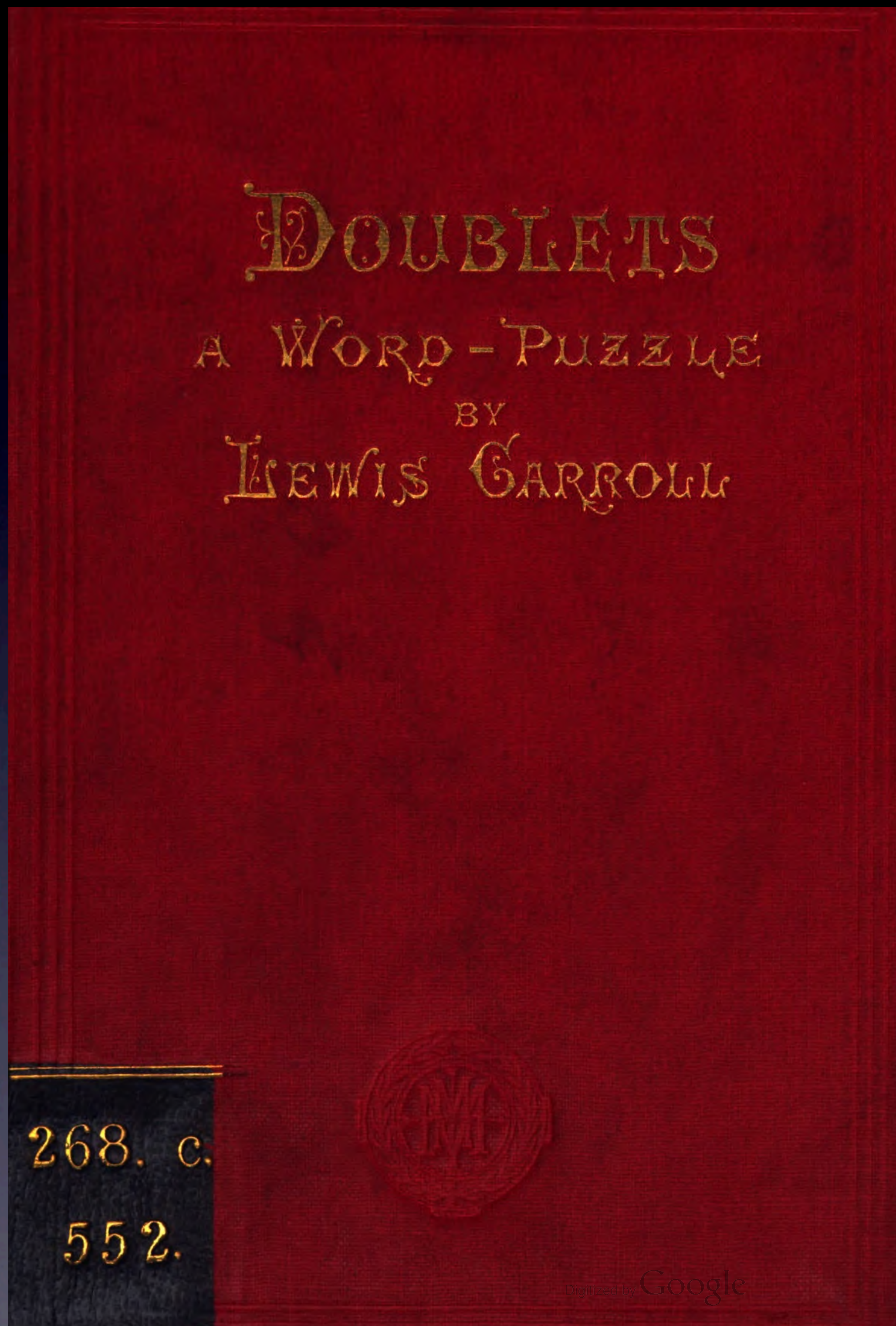
d states

n/2 verts

d^2 states

Must have $d^n = (d^2)^{n/2}$

Path through specified vertices

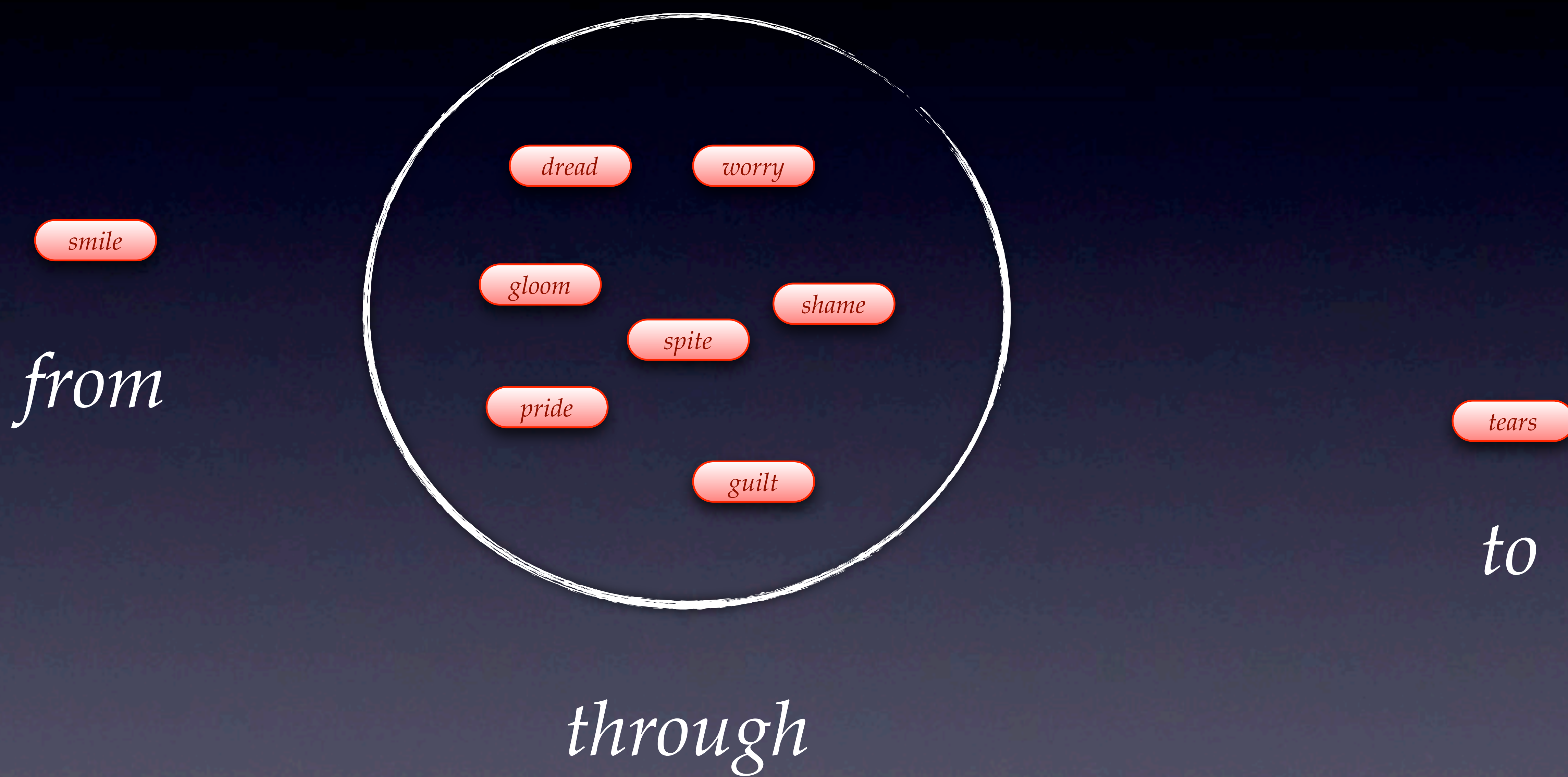


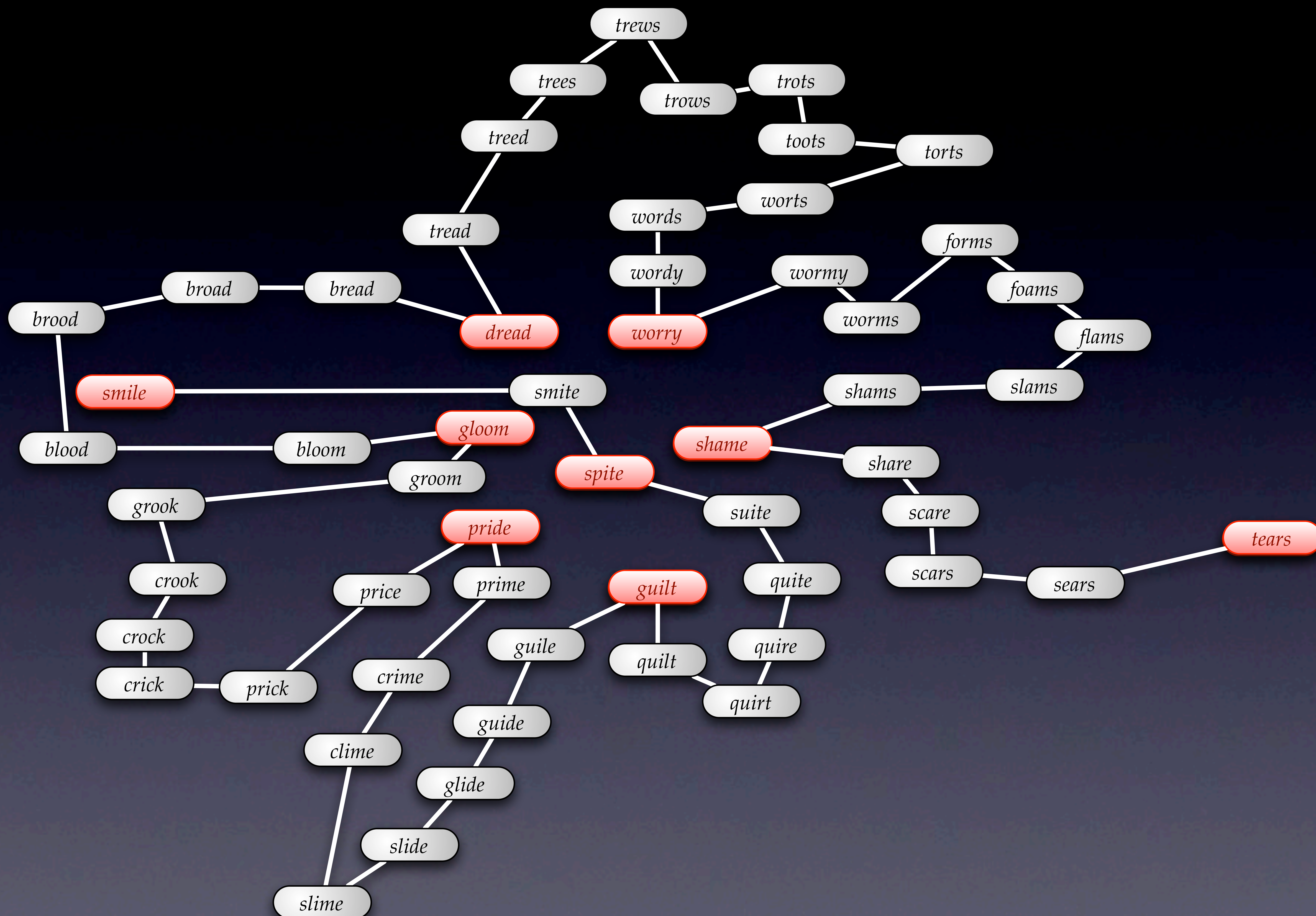
DEAR VANITY,—Just a year ago last Christmas, two young ladies —smarting under that sorest scourge of feminine humanity, the having "nothing to do"—besought me to send them "some riddles." But riddles I had none at hand, and therefore set myself to devise some

HEAD
h e a l
t e a l
t e l l
t a l l
TAIL

April 5.—Dip PEN into INK.
Touch CHIN with NOSE.
Change TEARS into SMILE.

Path through specified vertices




trews
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words
tread
forms
wordy
wormy
broad
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brood
dread
worry
worms
flams
smile
smite
shams
slams
gloom
blood
bloom
shame
groom
share
spite
grook
suite
scare
pride
tears
crook
price
prime
guilt
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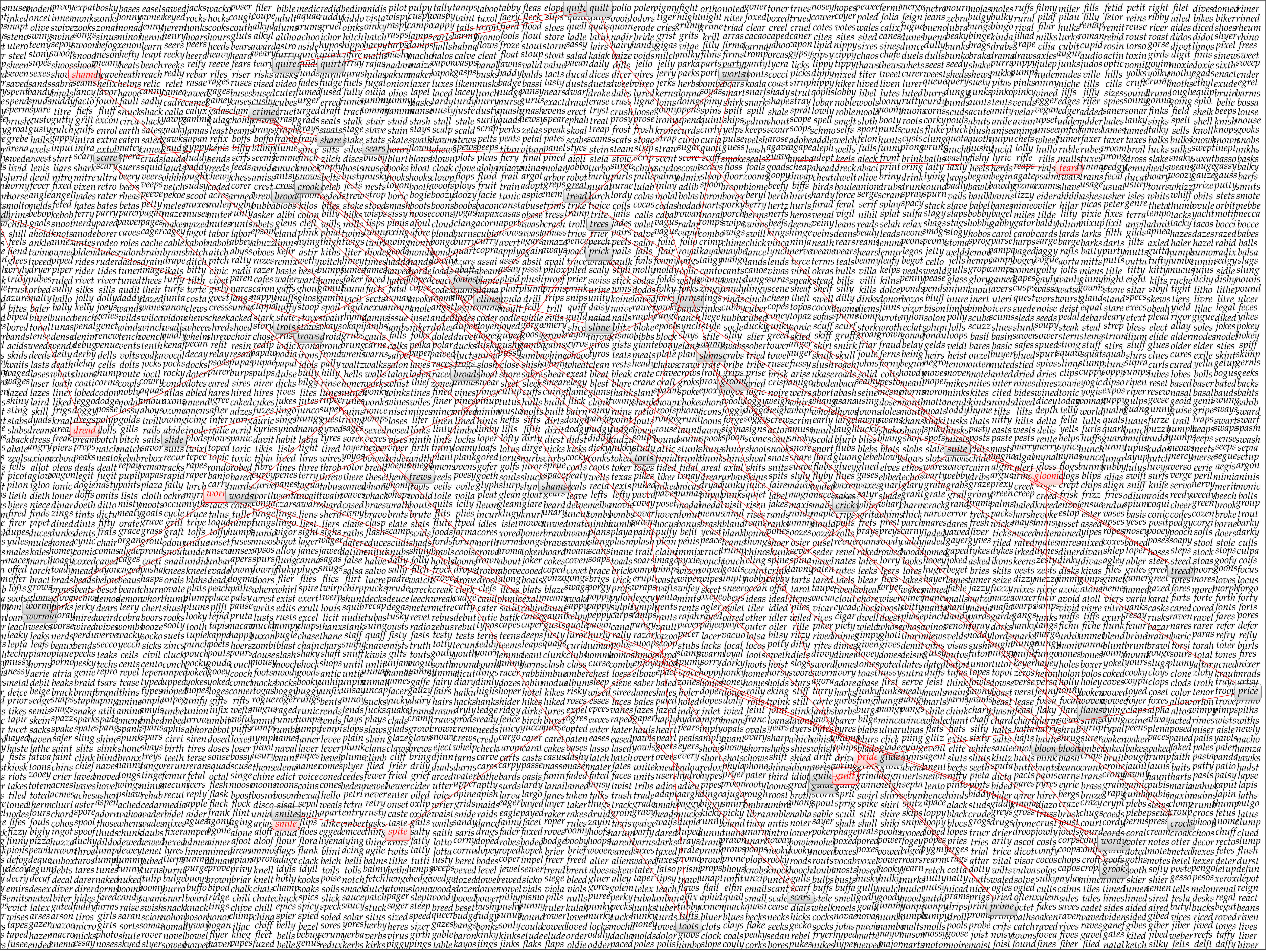
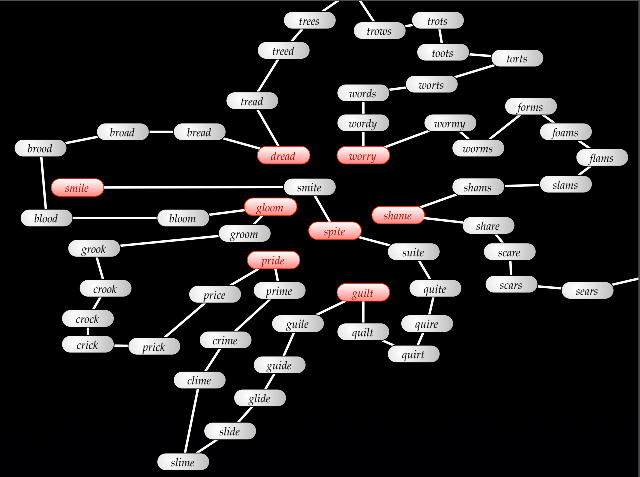
glide
slide
slime

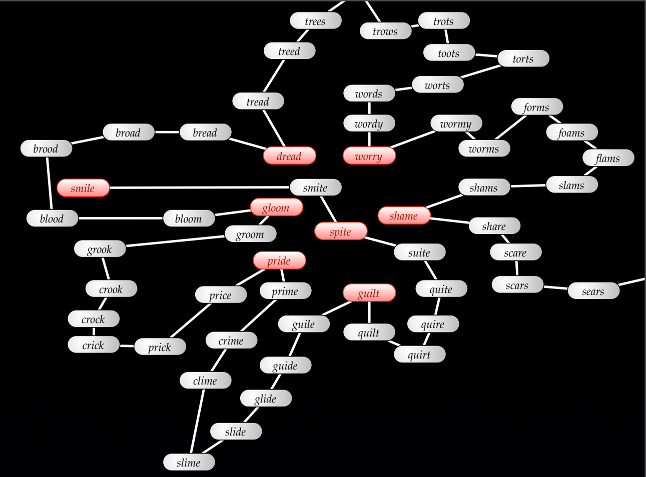


onsdag 12 oktober 11

k specified vertices, n vertices

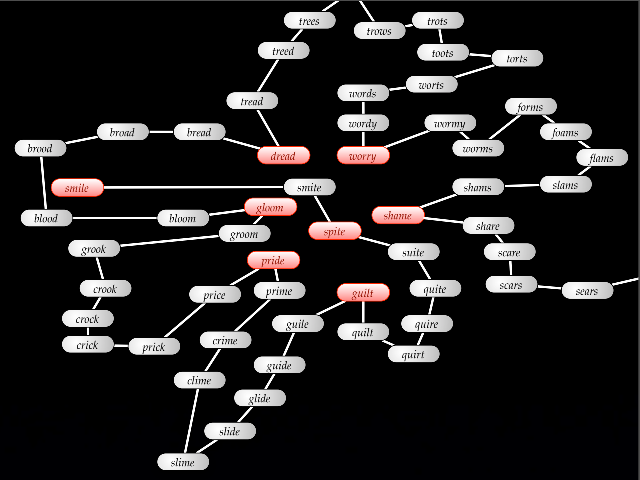


k specified vertices, n vertices



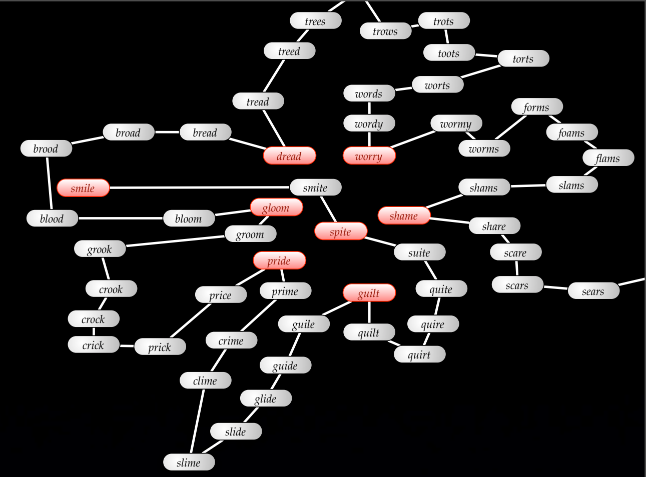
- $k=n$: Hamilton path

k specified vertices, n vertices

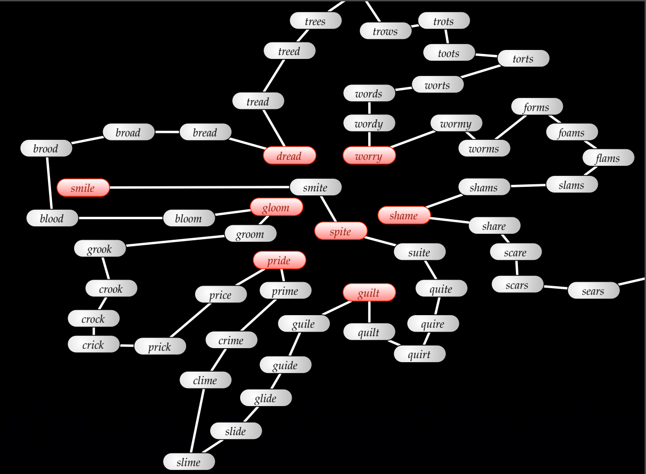


- $k=n$: Hamilton path
- no poly(k)-algorithm under P vs NP

k specified vertices, n vertices



- $k=n$: Hamilton path
- no poly(k)-algorithm under P vs NP
- no exp($o(k)$)-algorithm under ETH



k specified vertices, n vertices

- $k=n$: Hamilton path
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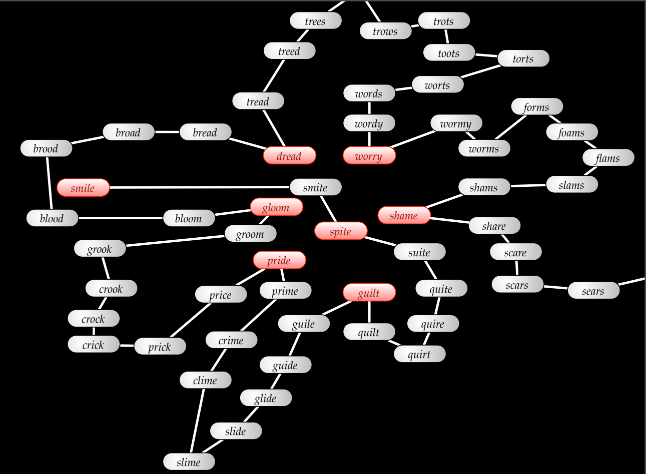
- no exp($o(k)$)-algorithm under ETH
- Brute force: $O(n!)$ (note: not n^k)



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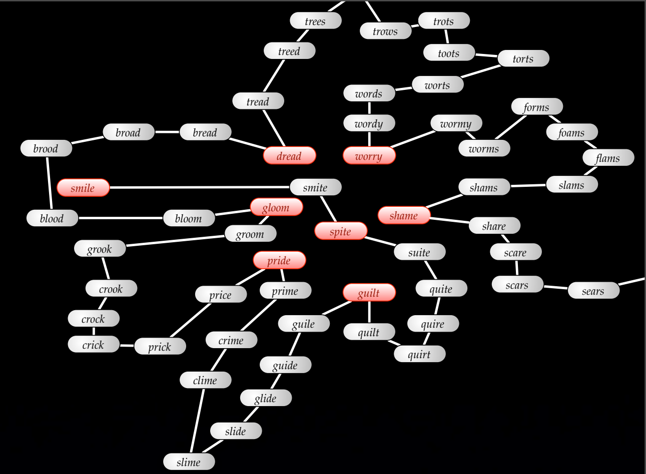
- Brute force: $O(n!)$ (note: not n^k)
- Disjoint paths: $f(k)\cdot\text{poly}(n)$

Robertson–Seymour

k specified vertices, n vertices



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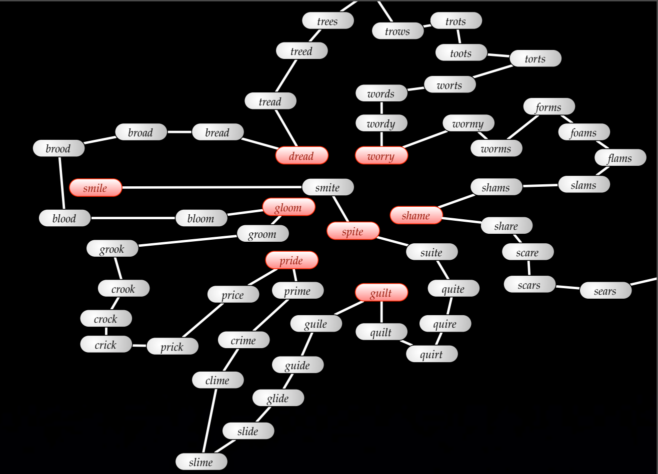
- Disjoint paths: $f(k)\cdot\text{poly}(n)$
- Algorithms for k=1,2,3.

Robertson–Seymour



k specified vertices, n vertices

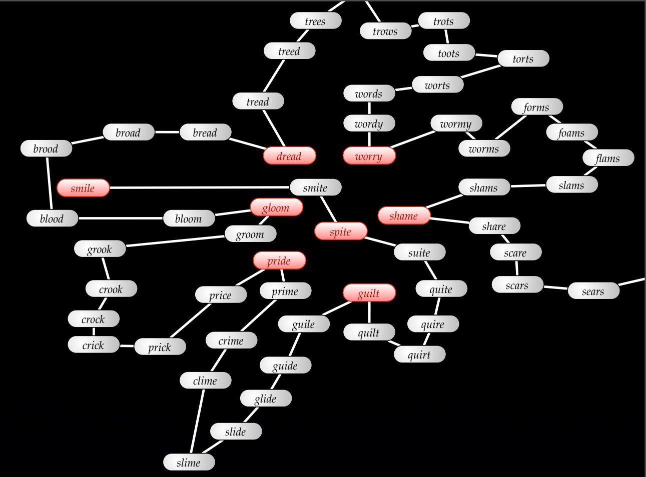
- $k=n$: Hamilton path
- no poly(k)-algorithm under P vs NP
- no exp($o(k)$)-algorithm under ETH

- Brute force: $O(n!)$ (note: not n^k)
- Disjoint paths: $f(k)\cdot\text{poly}(n)$
- Algorithms for k=1,2,3.
- Algorithm in $\exp(\exp(k^{10}))$

Robertson–Seymour

Kawarabayashi

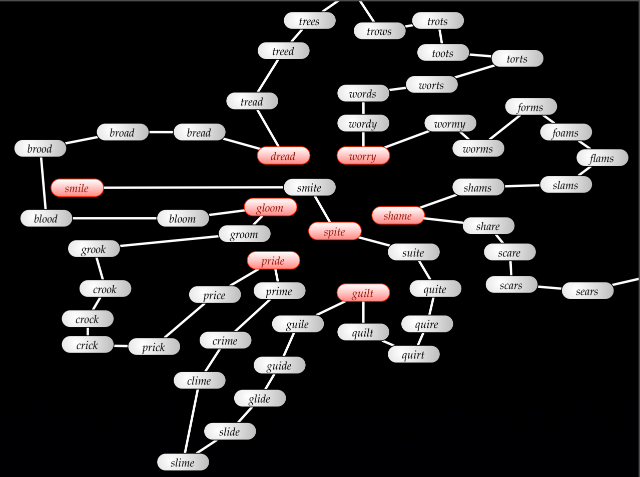
k specified vertices, n vertices



New result [SODA12]:
randomised algorithm in time
$\exp(k)\mathrm{poly}(n)$

Taslaman **Björklund**

k specified vertices, n vertices

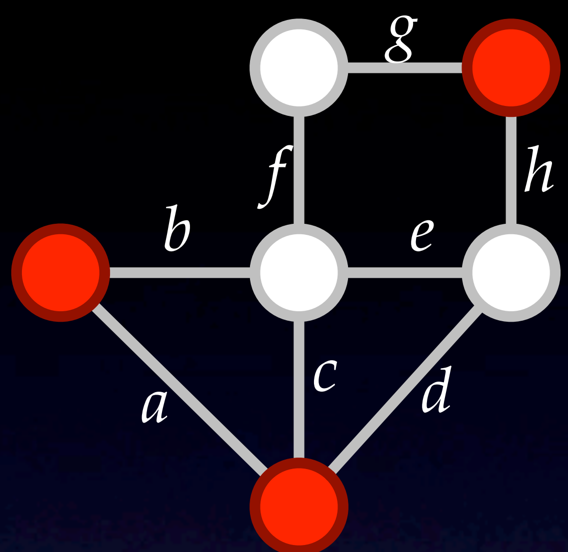


New result [SODA12]:
randomised algorithm in time
$\exp(k)\text{poly}(n)$

Thm: Shortest(!) cycle through k given vertices or edges in time $2^k\text{poly}(n)$ with exponentially small one-sided error.

Taslaman **Björklund**

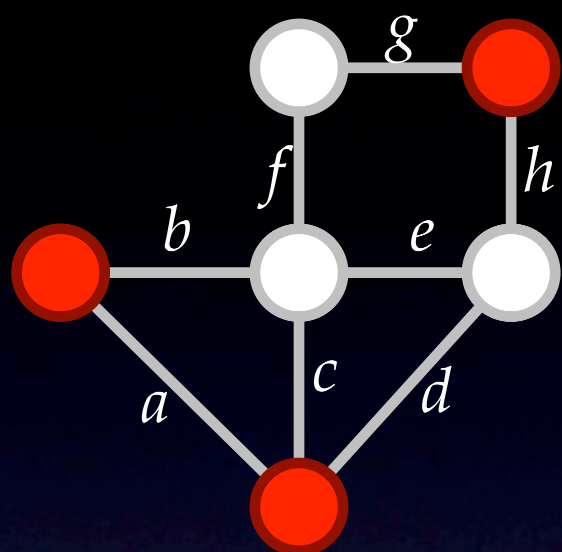
Trick: Look at Polynomials Instead



Koutis

Williams

Trick: Look at Polynomials Instead

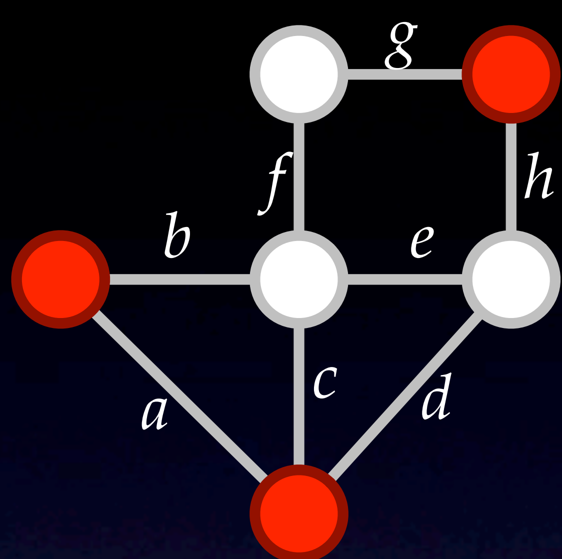


Koutis

Williams

Björklund *et al.*

Trick: Look at Polynomials Instead



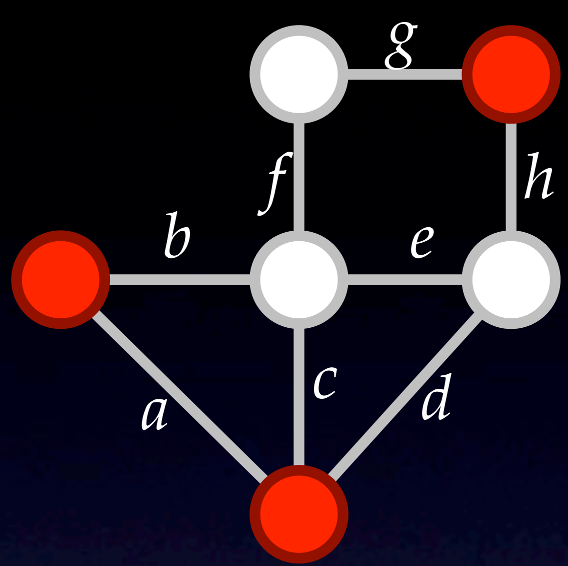
Koutis

Williams

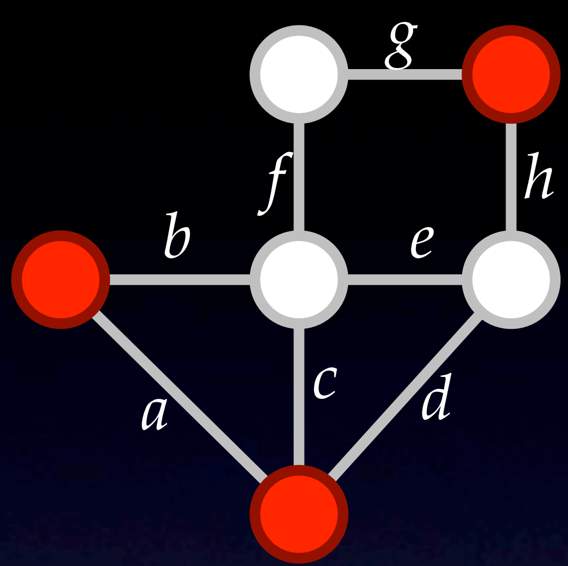
Björklund *et al.*

Tutte

Trick: Look at Polynomials Instead

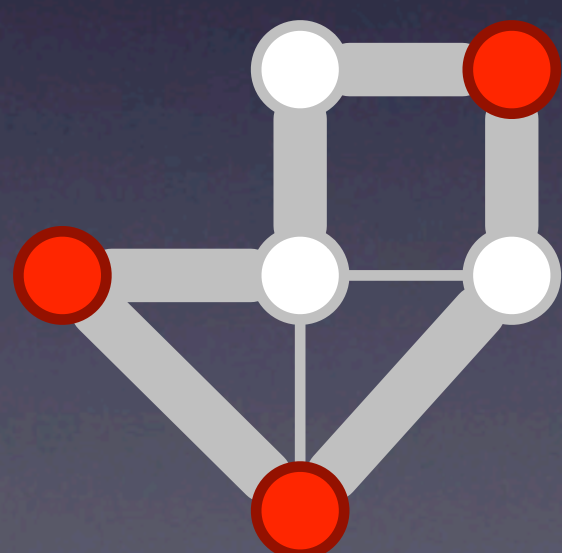


Trick: Look at Polynomials Instead

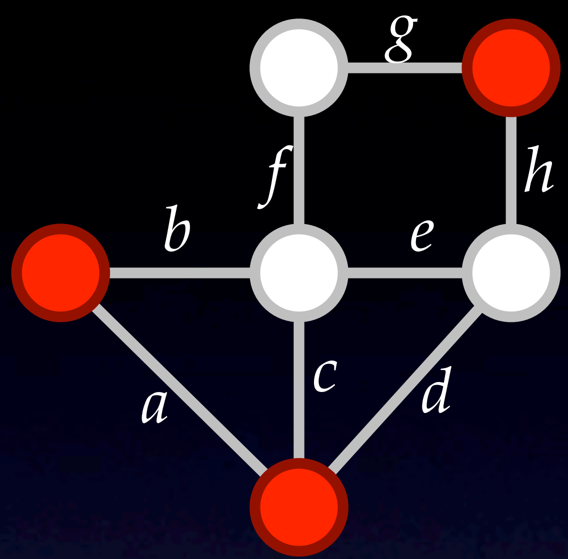


monomial for every walk

$a \cdot b \cdot f \cdot g \cdot h \cdot d$



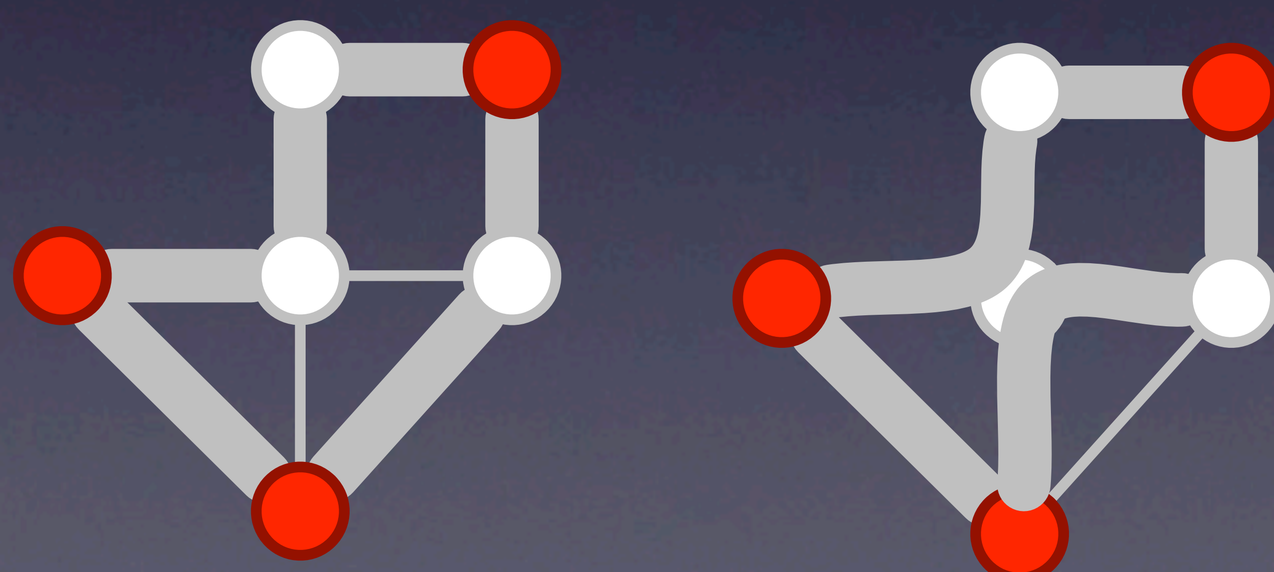
Trick: Look at Polynomials Instead



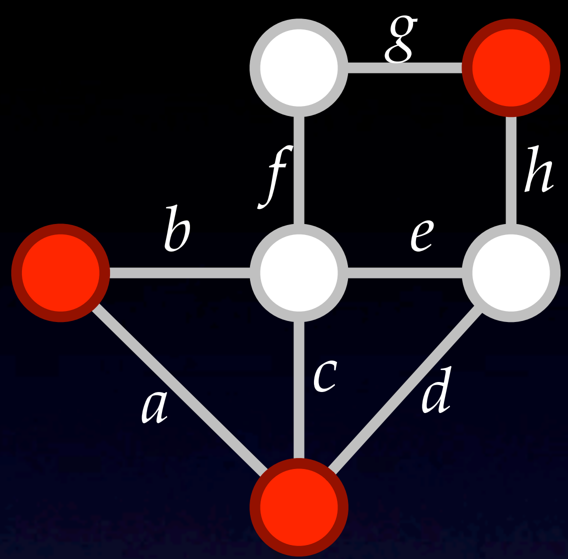
monomial for every walk

sum over all walks

$a \cdot b \cdot f \cdot g \cdot h \cdot d$ $\qquad$ $a \cdot b \cdot c \cdot e \cdot f \cdot g \cdot h$

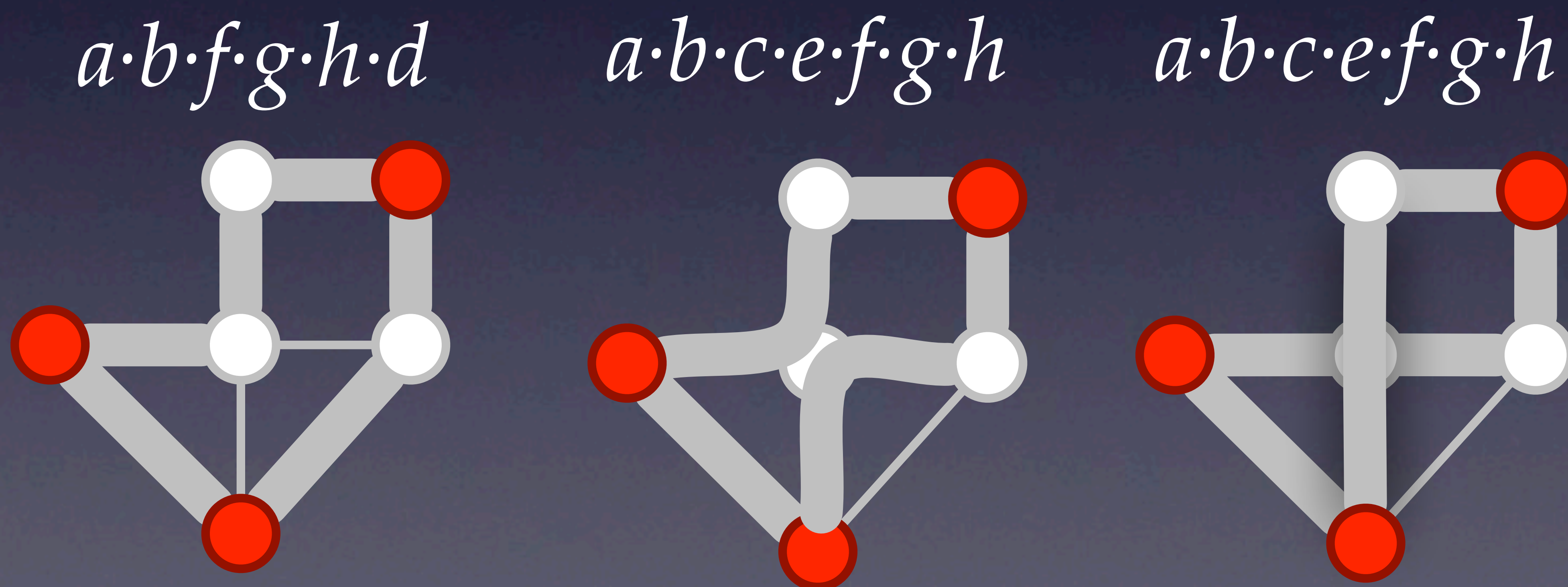


Trick: Look at Polynomials Instead

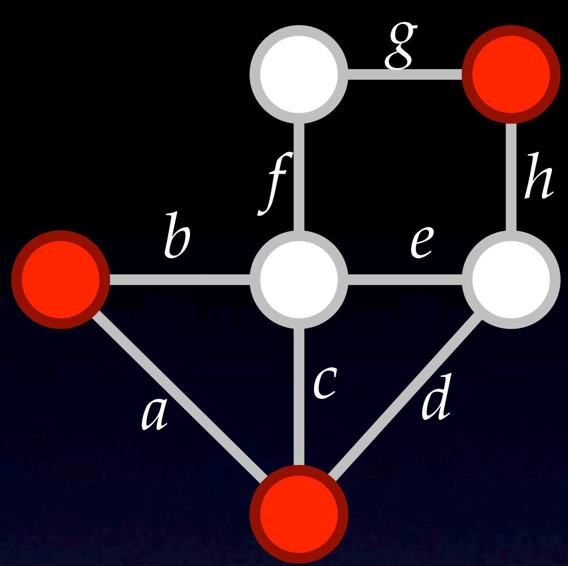


monomial for every walk

sum over all walks



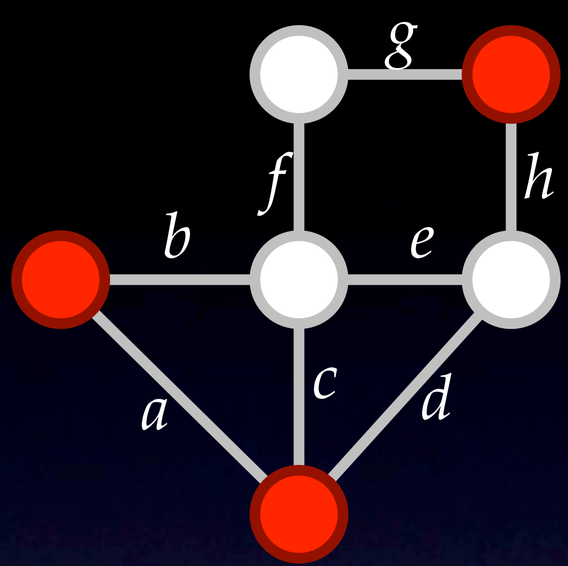
Trick: Look at Polynomials Instead



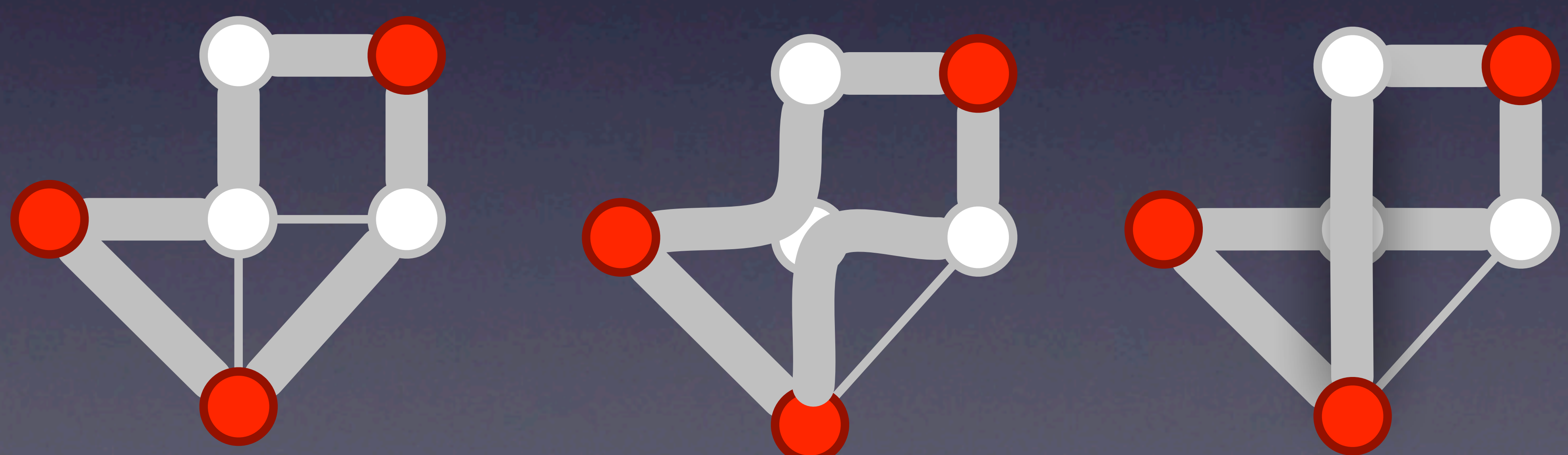
monomial for every walk

sum over all walks mod 2

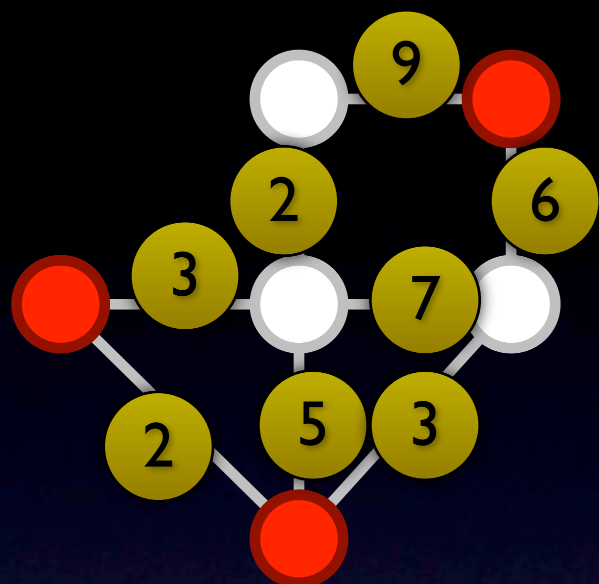
Trick: Look at Polynomials Instead



(Not really. Look at random numbers and *interpret* them as polynomial evaluations.)

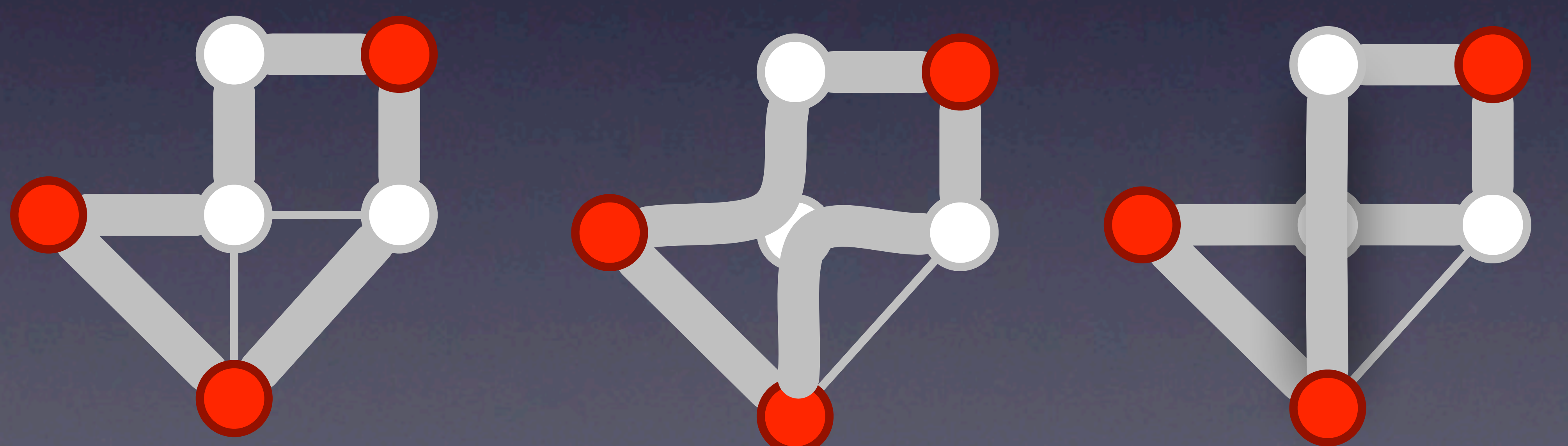


Trick: Look at Polynomials Instead

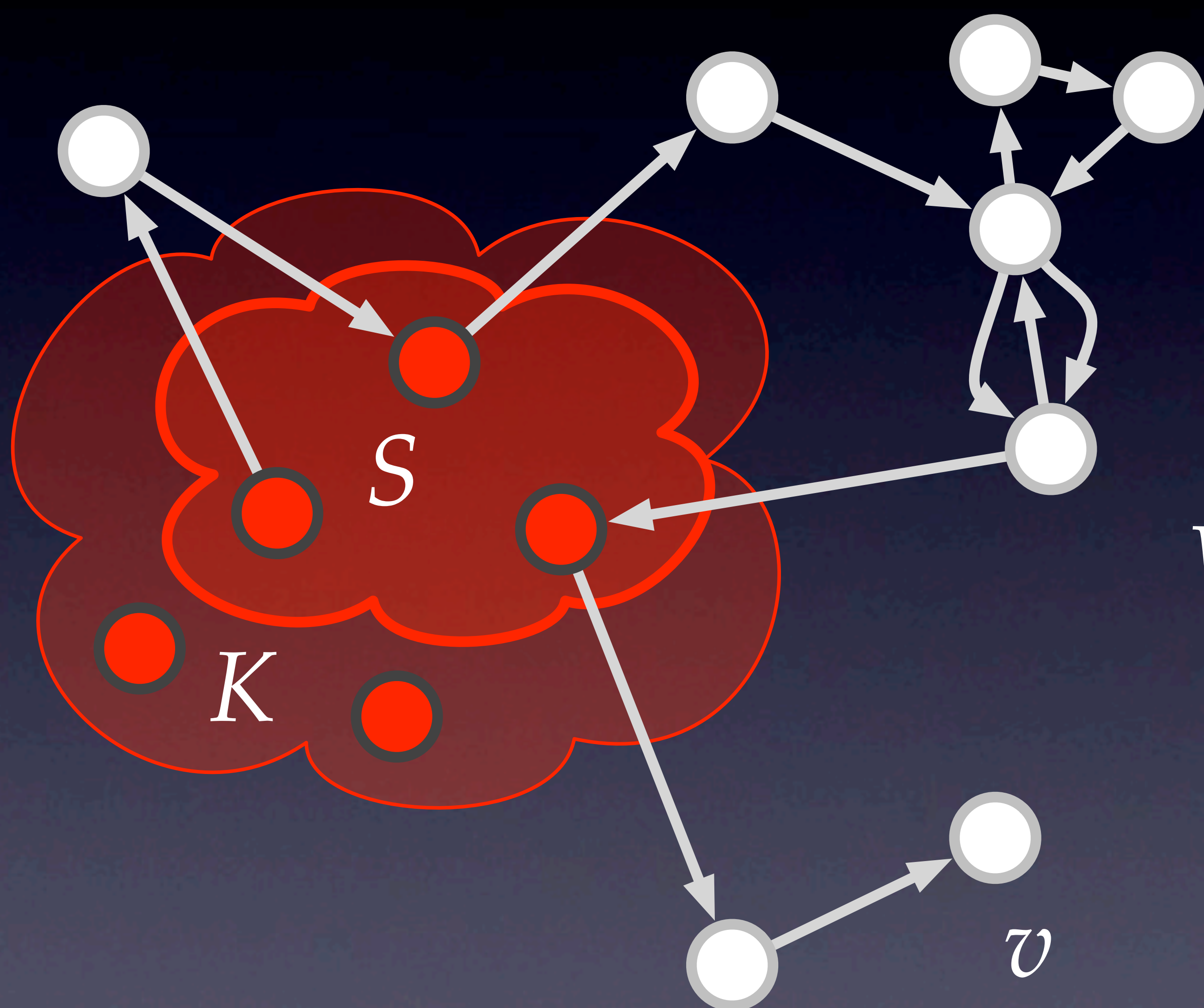


(Not really. Look at random numbers and *interpret* them as polynomial evaluations.)

$3{\cdot}2{\cdot}9{\cdot}6{\cdot}3{\cdot}2$ $\quad$ $3{\cdot}2{\cdot}9{\cdot}6{\cdot}7{\cdot}5{\cdot}2$ $\quad$ $3{\cdot}7{\cdot}6{\cdot}9{\cdot}2{\cdot}5{\cdot}2$



Constructing all Walks: Dynamic Programming for Sequencing Problems



K = specifed vertices

$S \subseteq K$

W$(r,S,v) =$ *walks s.t.*

length: r

end in v

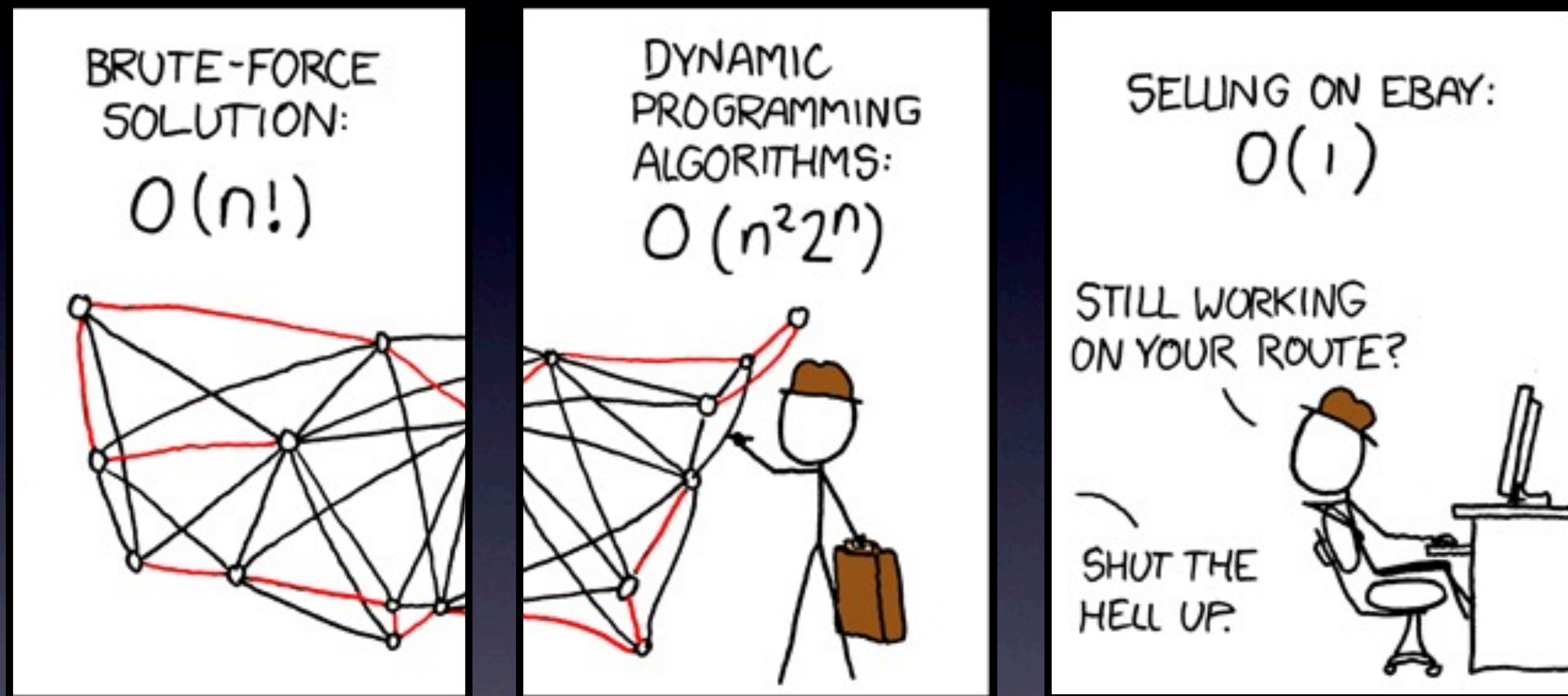
visit all in S exactly once

no other in K

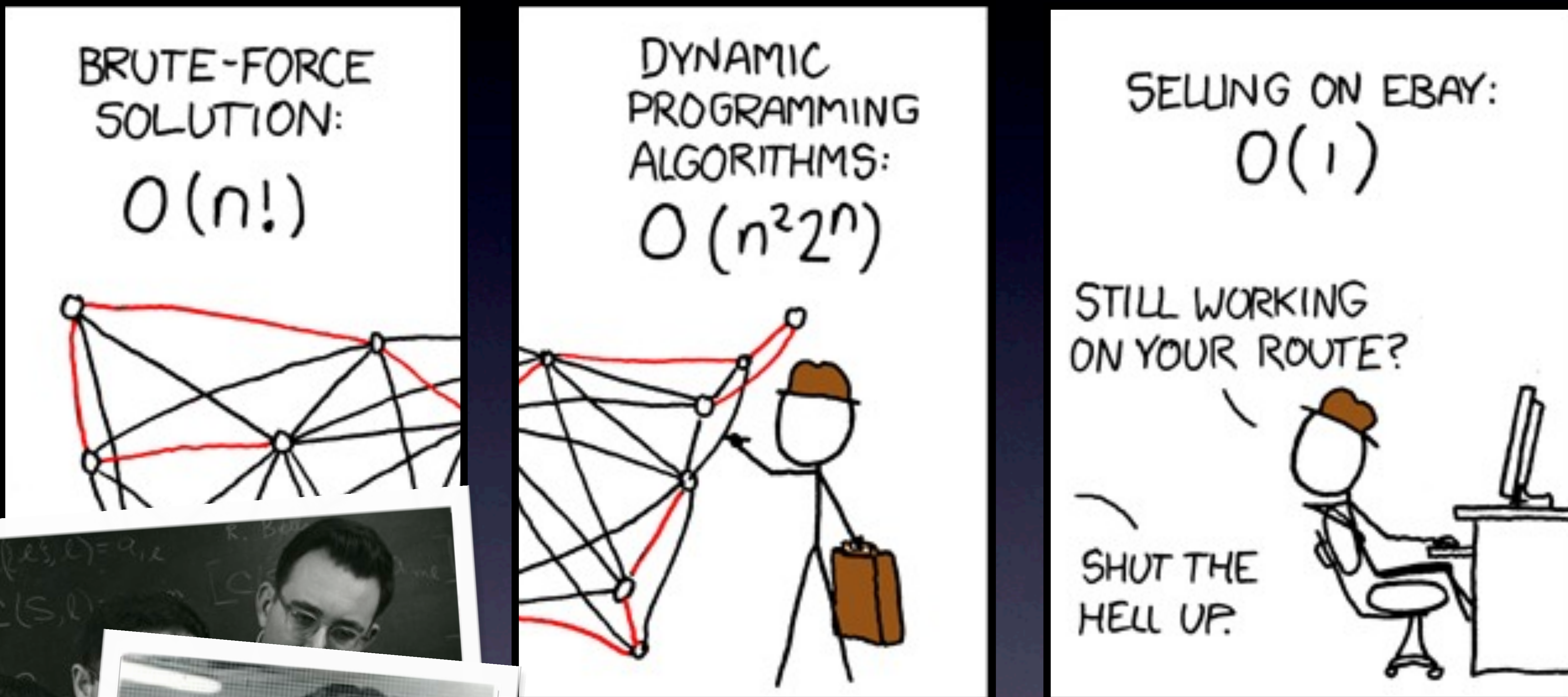
$$W(r,S,v) = \begin{cases} \bigcup_{uv \in E} W(r-1, S, u) & v \notin S \\ \bigcup_{uv \in E} W(r-1, S-v, u) & v \in S \end{cases}$$

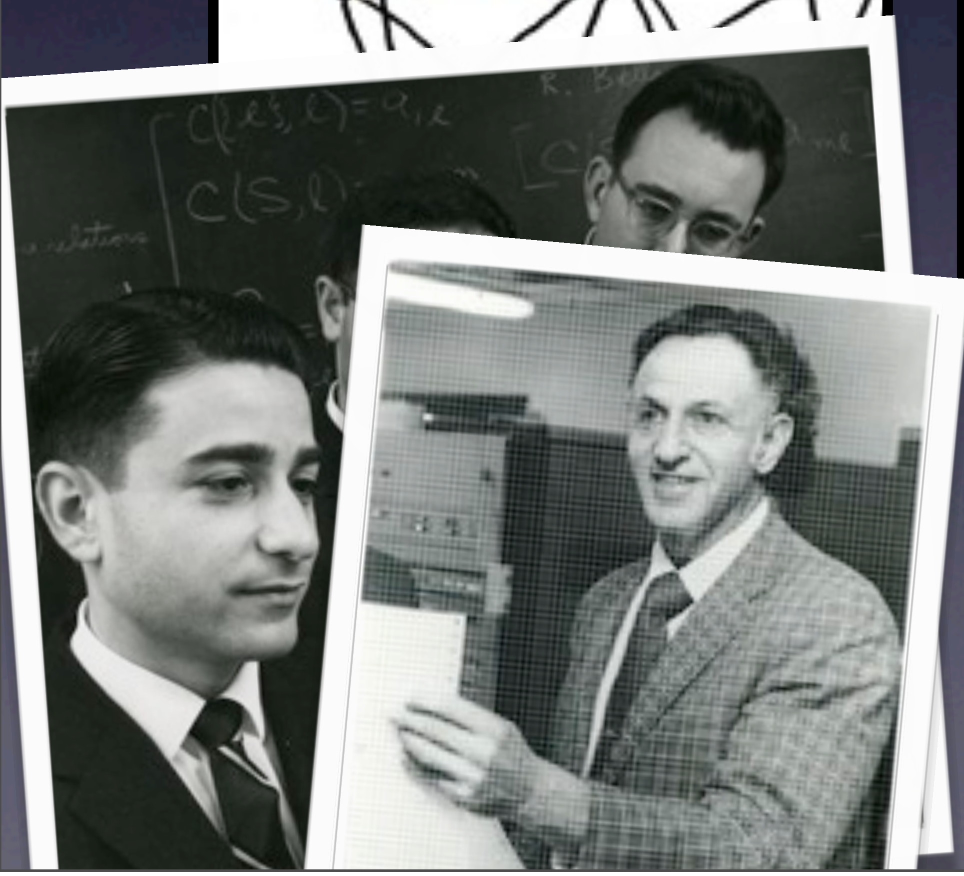
Time: 2^Kpoly(n)

Constructing all Walks:
Dynamic Programming for Sequencing Problems



Constructing all Walks: Dynamic Programming for Sequencing Problems

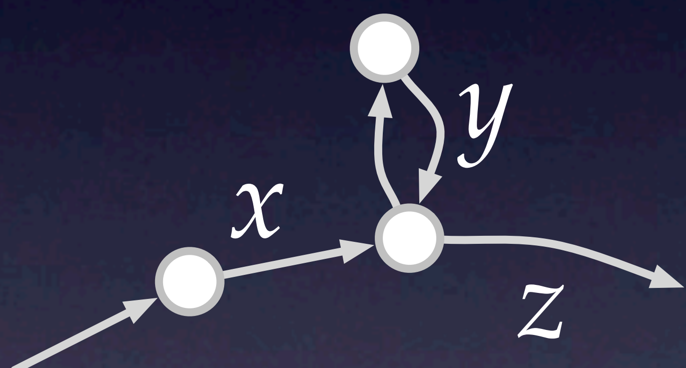




Held–Karp

Bellman

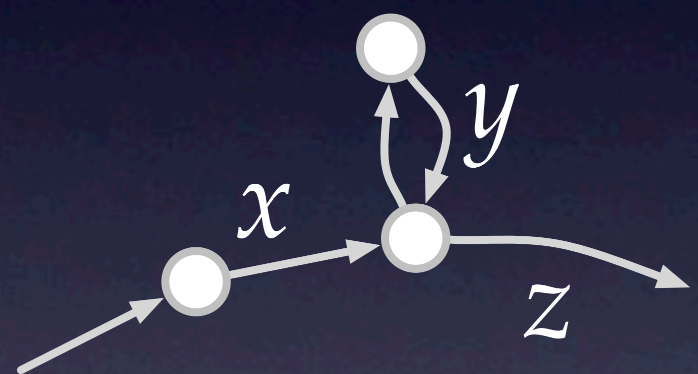
Some pitfalls…



does not cancel

xy^2z

Some pitfalls…

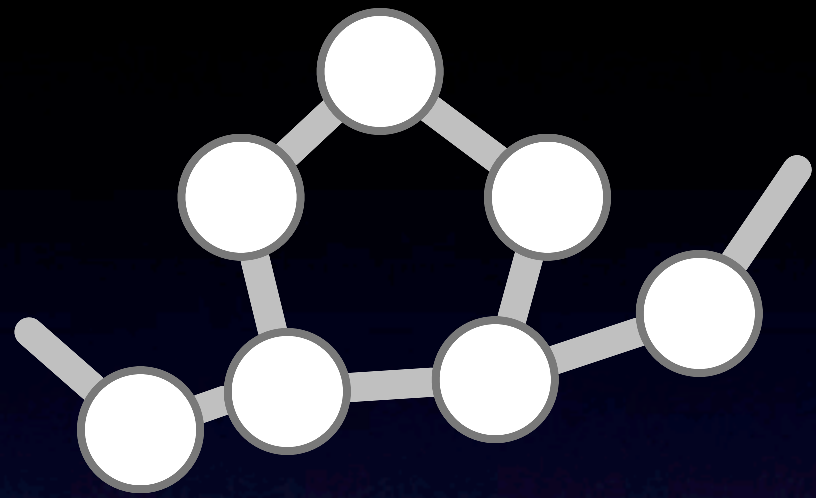


does not cancel

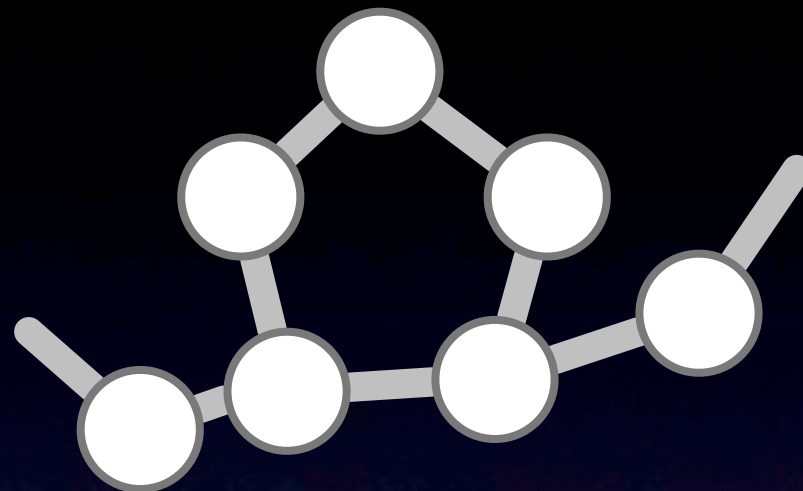
xy^2z

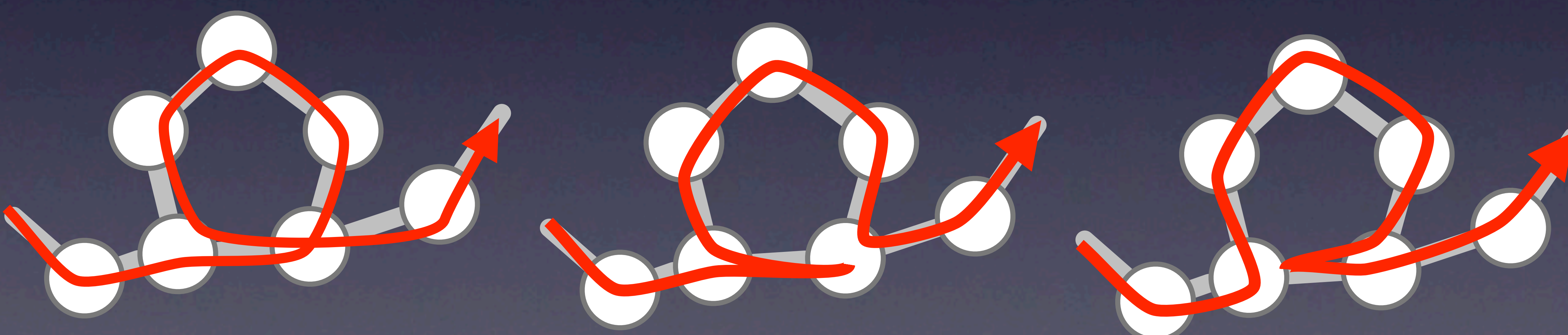
(solved in the dynamic program: just avoid “digons”)

Some pitfalls…

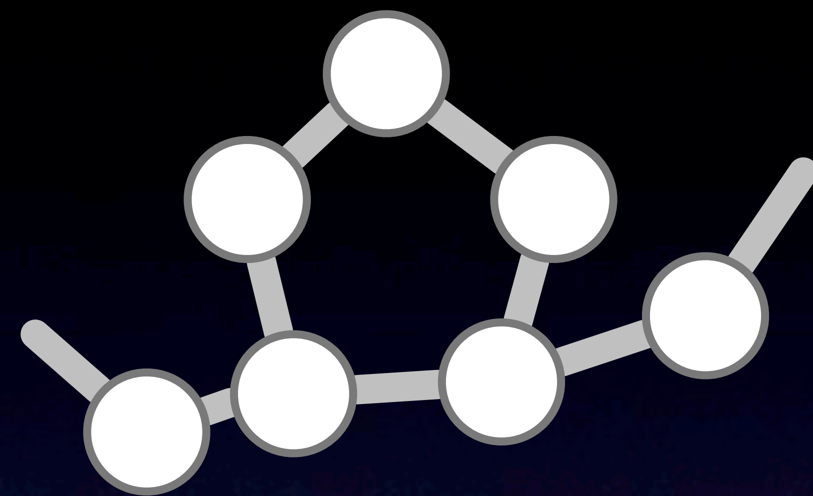


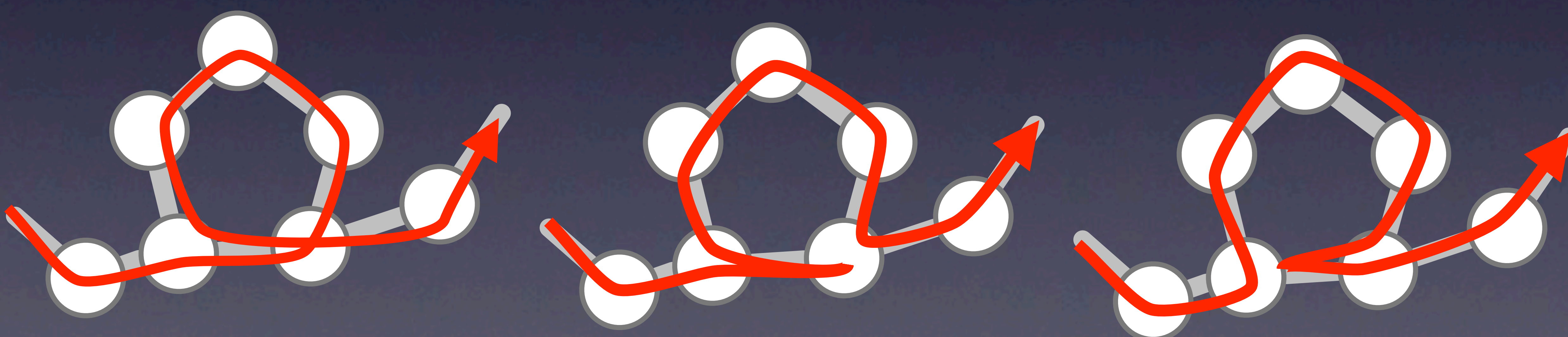
Some pitfalls…





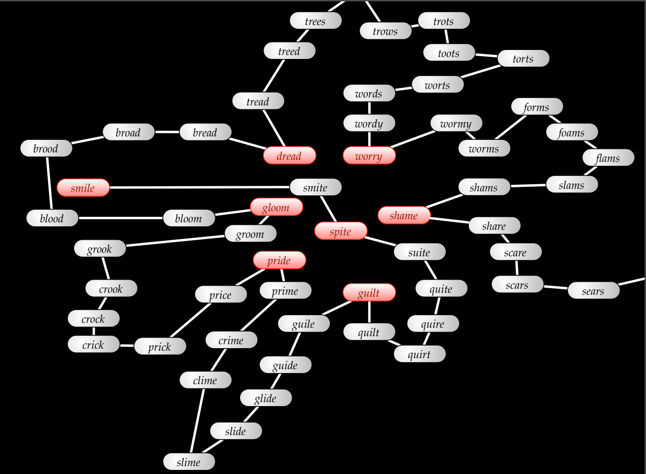
Some pitfalls…





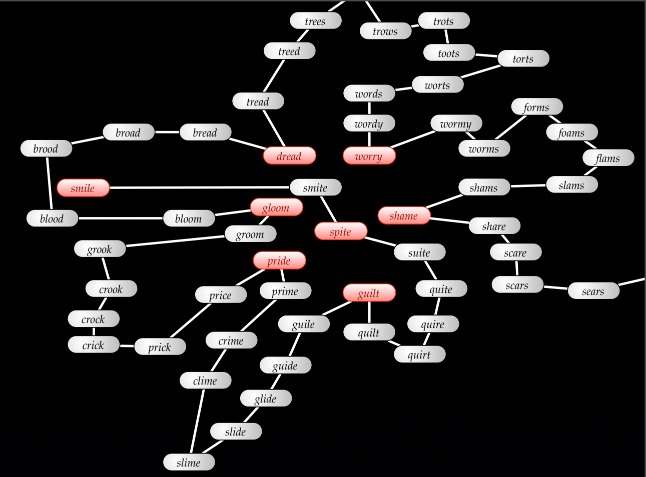
(solve shorter lengths first)

k specified vertices, n vertices



1. Associate random value from GF(2^q) to each edge
2. Use dynamic programming to count the contribution of all sufficiently well-behaved walks
3. Return "Found one!" if the result is nonzero

k specified vertices, n vertices



Theorem: *Shortest cycle through k given vertices or edges in time* $2^k\text{poly}(n)$ *with exponentially small one-sided error.*

1. Associate random value from GF(2^q) to each edge
2. Use dynamic programming to count the contribution of all sufficiently well-behaved walks
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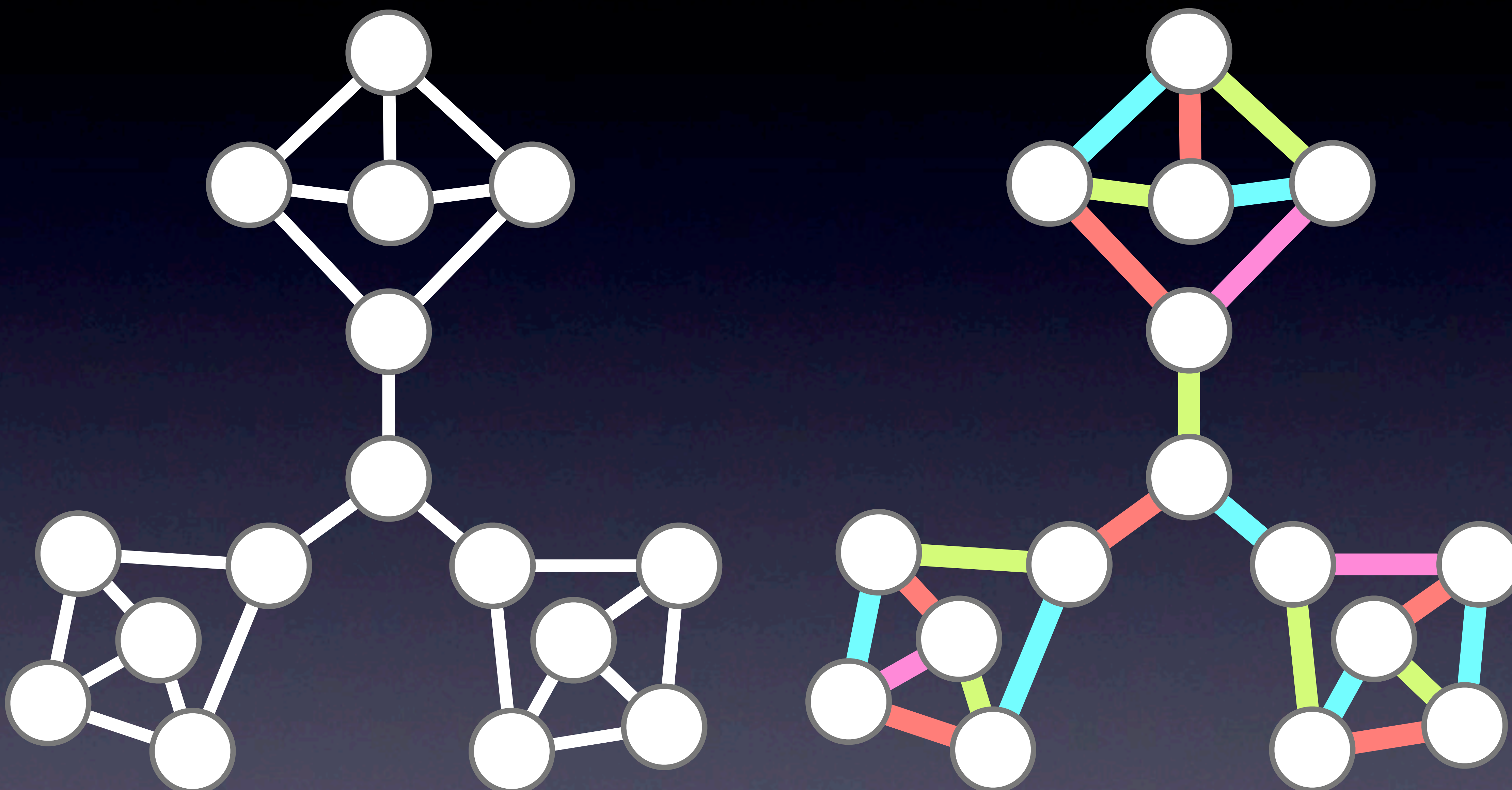
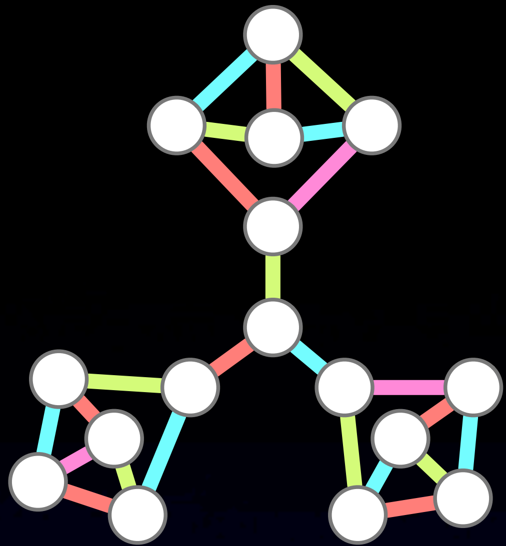
Edge Colouring

Edge Colouring





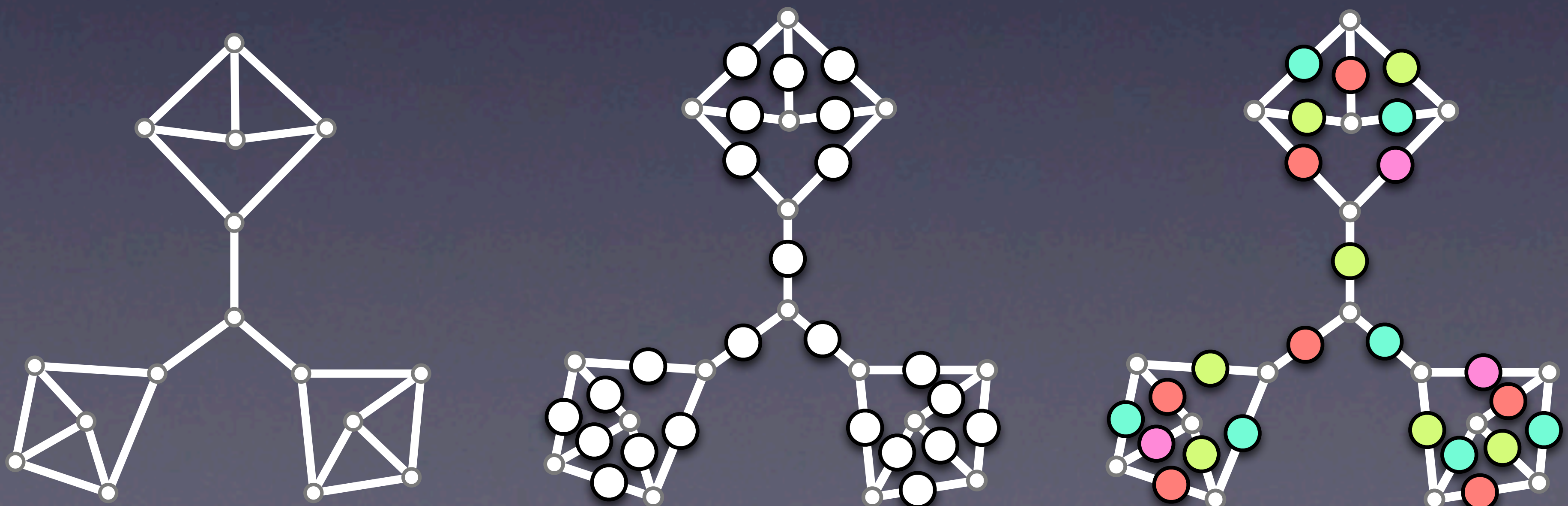
k: # colours

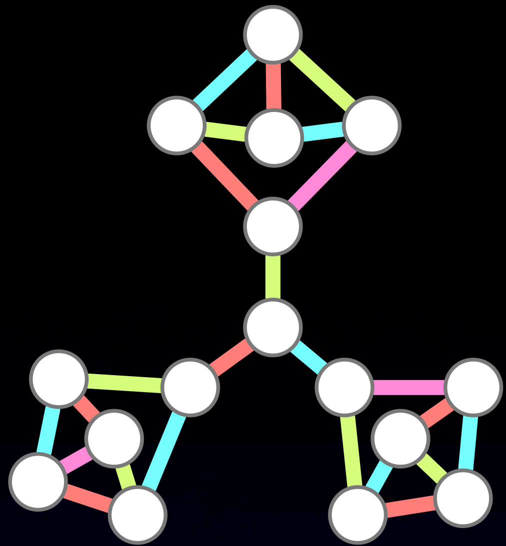
d: degree

Vizing: $k = d$ or $k = d$+1

Brute force: check all d^m possibilities

Vertex colour the line graph: time 2^m=$2^{nd/2}$





k: # colours

d: degree

Brute force	d^m
Vertex colour the line graph	$2^m = 2^{nd/2}$
"Narrow sieves" [*BHKK*]	$2^{n(d-1)/2}$

Under ETH: not in $\exp(o(n))$

Edge Colouring takes

- ☐ $\exp(n)$
- ☐ $d^n = \exp(n \log d)$
- ☐ $\exp(m) = \exp(nd)$

Tak fordi I kom