Grothendieck inequalities for semidefinite programs with rank constraints

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The plain-vanilla-flavor SDP problem

Given: symmetric matrix $A \in \mathbb{R}^{n \times n}$ with $\mathrm{diag}(A) = 0$

Find: unit vectors $x_1, \ldots, x_n$ that maximize the sum

$$\sum_{1 \le i < j \le n} A(i,j) \cdot \langle x_i, x_j \rangle$$

Can be solved in poly-time

SDPs with rank* constraint

Adding a little more structure...

Given: symmetric matrix $A \in \mathbb{R}^{n \times n}$ with $\mathrm{diag}(A) = 0$ and positive integer r

Find: r-dimensional unit vectors $x_1, \ldots, x_n$ that maximize the sum

$$\sum_{1 \leq i < j \leq n} A(i,j) \cdot \langle x_i, x_j \rangle$$

Denote this problem by SDP_r and its optimum by $\mathrm{SDP}_r(A)$

SDP_∞ is the *SDP relaxation* of SDP_r: *"drop the rank constraint"*

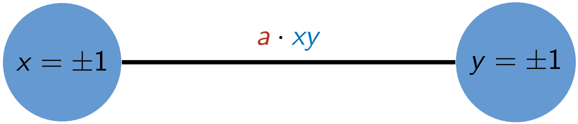
*The word *rank* appears because the matrix $X(i,j) = \langle x_i, x_j \rangle$ has rank r

A tiny example: $n = 2$

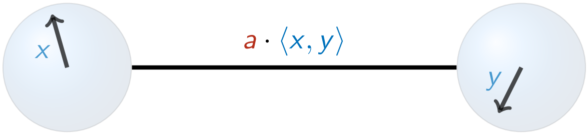
Given: $a \in \mathbb{R}$ and $r \in \mathbb{N}$

Find: $x, y \in S^{r-1}$ that maximize $a \cdot \langle x, y \rangle$

The rank-1 case has a combinatorial nature



Higher ranks have a more geometric flavor

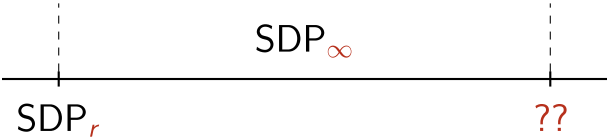


Applications of SDPs with rank constraint

- Combinatorial cases (rank-1):
  - MAX CUT
  - cut-norm of a matrix
  - statistical physics (Ising spin glasses)
  - communication complexity
- Geometrical cases (ranks ≥ 2):
  - quantum information theory
  - statistical physics (planar and Heisenberg spin glasses)

MAIN QUESTION

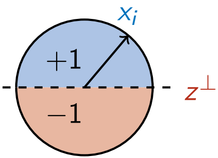
How close are $\mathrm{SDP}_{\infty}(A)$ *and* $\mathrm{SDP}_{r}(A)$*?*



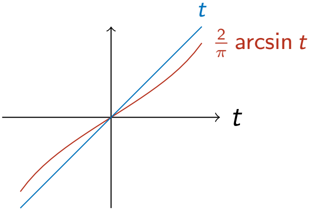
Inapproximability results are known for all ranks $r \geq 1$

“Hyperplane rounding” does not work

- Obvious strategy to approximate SDP_1 by SDP_∞
  1. Solve SDP_∞, get vectors $x_1, \ldots, x_n \in S^{n-1}$
  2. Sample vector $z \in \mathbb{R}^n$ with iid $N(0,1)$ entries
  3. Round: Set $y_i = \mathrm{sign}\langle x_i, z \rangle$



- Grothendieck identity: $\mathbb{E}_z[y_i y_j] = \frac{2}{\pi} \arcsin(\langle x_i, x_j \rangle)$

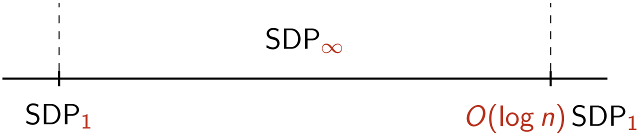


- Coefficients $A(i,j)$ of $y_i y_j$ give bad cancellations

Approximation results for the rank-1 case

Positive result

- [N98, NRT99, Meg01, CW04]: $O(\log n)$-approximation

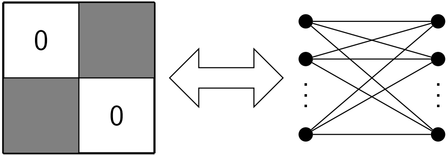


Negative results

- [KO'D09]: Matching $\Omega(\log n)$ lower bound
- [ABKHS05, KO'D09]: Hardness-of-approximation results

Better results hold for "bipartite matrices"...

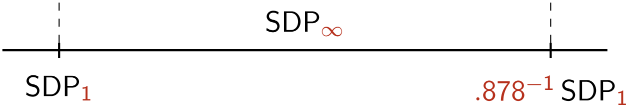
Matrices with bipartite support graph



- For graph $G = (V, E)$ and $W = \mathrm{diag}(\deg(V)) - \mathrm{Adj}(G)$,

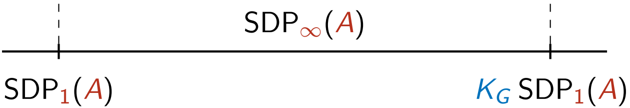
$$\mathrm{SDP}_1\left(\begin{bmatrix} 0 & W \\ W^T & 0 \end{bmatrix}\right) = 4|\text{MAX CUT}(\mathrm{G})|$$

- [GW95]: .878-approximation for these types of matrices



Grothendieck's inequality

- [AN04]: $O(1)$-approximation of $\mathrm{SDP}_1(A)$ for bipartite A
- Based on an algorithmic proof of *Grothendieck's inequality:*



  for universal <u>constant</u> K_G and bipartite A [Grothendieck53]

- Exact value of K_G: unknown
- [Krivine79]: $K_G \leq 1.78\ldots$, [BMMN11]: $K_G < 1.78\ldots$
- [RS09]: Assuming UGC, $\nexists$ $(K_G - \delta)$-approximation for $\delta > 0$

Other support graphs? Higher ranks?

- Big contrast between complete and bipartite support graphs
- *Better approximation results for other support graphs??*
- *What about higher ranks??*

The graphical Grothendieck problem with rank-r contstraint

Given: graph $G = (V, E)$,
symmetric matrix A with support graph G
and positive integer r

Find: r-dimensional unit vectors x_i
that maximize the sum

$$\sum_{i,j} A(i,j) \cdot \langle x_i, x_j \rangle$$

Application: spin glasses

Model of interacting particles introduced by Stanley (1968)

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Spherical Model as the Limit of Infinite Spin Dimensionality

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(Received 20 April 1968)

The Berlin-Kac spherical model (or "spherical approximation to the Ising model")

$$\mathcal{H}^{\mathrm{SM}} = -J \sum_{\langle ij \rangle} \mu_i \mu_j, \quad \text{with} \quad \sum_{j=1}^{N} u_j^2 = N,$$

is found to be equivalent to the $\nu \to \infty$ limit of the Hamiltonian

$$\mathcal{H}^{(\nu)} = -J \sum_{\langle ij \rangle} \mathbf{S}_i^{(\nu)} \cdot \mathbf{S}_j^{(\nu)},$$

where $\mathbf{S}_j^{(\nu)}$ are isotropically interacting ν-dimensional classical spins.

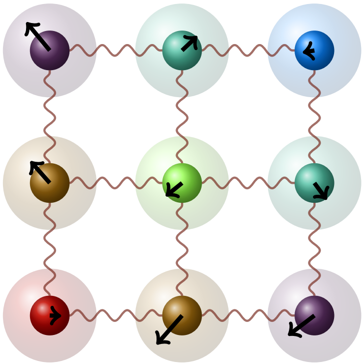
I. INTRODUCTION

THE Berlin-Kac spherical model[1] has received considerable attention, particularly because it is

Although it is perhaps not generally appreciated, it seems clear that $\mathcal{H}^{(\nu)}$ reduces to the $S=\frac{1}{2}$ Ising, classical planar, and classical Heisenberg models for $\nu = 1$, 2, and 3 respectively.

Geometric instances: spin glasses

Particles are located at vertices of an interaction graph



Particles are unit vectors

- 1D = Ising model
- 2D = planar model
- 3D = Heisenberg model

Edge weights $W : E \to \mathbb{R}$ give their interaction strength

Problem: compute the *ground state* of the total system:

$$-\max \sum_{\{u,v\} \in E} W(u,v) \langle x_u, x_v \rangle$$

Approximation results

$K(r, G) = \max \frac{\mathrm{SDP}_\infty(A)}{\mathrm{SDP}_r(A)}$ over matrices A with support graph G

upper bounds on $K(r, G)$ ("integrality gaps")

rank r \ $\chi(G)$	2 (bipartite)	3 (triparite)	4…	n (complete)
	*	*	*	$O\left(\frac{\log \chi}{r}\right)$*
4	1.216… *	2.309… *	*	
3	1.280… *	2.412… *	*	
2	1.404… *	2.621… *	*	
1	$< 1.782\ldots$ [BMMN11]	3.264… *	*	$O(\log \chi)$[AMMN06]

*OUR RESULTS

Proof sketch for $\chi(G) = 2$, rank ≥ 1

- Want to show: $\mathrm{SDP}_r(A) \geq c\,\mathrm{SDP}_\infty(A)$ for bipartite A
- Transform optimal SDP_∞ vectors x_i into <u>r-dimensional</u> y_i s.t.

$$\langle y_i, y_j \rangle = c\,\langle x_i, x_j \rangle$$

- y_i are *feasible* for SDP_r
- they give value

$$\sum A(i,j)\,\langle y_i, y_j \rangle = c\,\mathrm{SDP}_\infty(A)$$

- Hence, $\mathrm{SDP}_r(A) \geq c\,\mathrm{SDP}_\infty(A)$
- $\rightsquigarrow K(r, G) \leq 1/c$

How to transform the SDP solution vectors??

Random "rounding" and a generalized Grotendieck identity

- Sample $Z \in \mathbb{R}^{r \times n}$ with iid $N(0,1)$ entries
- For optimal SDP_∞ vector x, set $y = Zx/\|Zx\|_2$
- What we would like to hold: $\mathbb{E}_Z[\langle y_i, y_j \rangle] = c\langle x_i, x_j \rangle$

Theorem. $\mathbb{E}_Z[\langle y_i, y_j \rangle] = E_r\big(\langle x_i, x_j \rangle\big)$

$$= \gamma(r) \times \Big(\langle x_i, x_j \rangle + \frac{1}{2(r+2)} \langle x_i, x_j \rangle^3 + \frac{9}{8(r+2)(r+4)} \langle x_i, x_j \rangle^5 + \frac{225}{48(r+2)(r+4)(r+6)} \langle x_i, x_j \rangle^7 + \frac{11025}{384(r+2)(r+4)(r+6)(r+8)} \langle x_i, x_j \rangle^9 + \frac{893025}{3840(r+2)(r+4)(r+6)(r+8)(r+10)} \langle x_i, x_j \rangle^{11} + \cdots \Big)$$

- Grothendieck's identity: $E_1 = \frac{2}{\pi} \arcsin$

Krivine's embedding technique

- Embed the SDP_∞ vectors x_i before rounding
- To embed, use the *inverse* of E_r

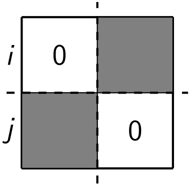
$$E_r^{-1}(t) = \alpha_1 t + \alpha_2 t^2 + \cdots$$

- Set $S(x) = [\quad \sqrt{c|\alpha_1|}x \quad , \quad \sqrt{c^2|\alpha_2|}x^{\otimes 2} \quad , \ldots]$
- $T(x) = [\text{sign}(\alpha_1)\sqrt{c|\alpha_1|}x \, , \, \text{sign}(\alpha_2)\sqrt{c^2|\alpha_2|}x^{\otimes 2} \, , \ldots]$
- Inner product of $S(x)$ and $T(y)$ *inverts* E_r

$$\langle S(x), S(y) \rangle = E_r^{-1}(c\langle x, y \rangle)$$

Putting things together: *"Embed, then round"*

1. Get optimal vectors x_i for SDP_∞
2. Krivine embedding
   - For "left" index i set $\tilde{x}_i = S(x_i)$
   - For "right" index j set $\tilde{x}_j = T(x_i)$
3. "Round"
   - Sample $Z \sim N(0,1)^{r \times n}$
   - set $y_i = Z\tilde{x}_i / \|Z\tilde{x}_i\|_2$
4. We have



$$
\begin{aligned}
\mathbb{E}_Z[\langle y_i, y_j \rangle] &= E_r(\langle \tilde{x}_i, \tilde{x}_j \rangle) \quad \text{Grothenedieck identity} \\
&= E_r\left(E_r^{-1}(c\langle x_i, x_j \rangle)\right) \quad \text{Krivine's trick} \\
&= c\langle x_i, x_j \rangle
\end{aligned}
$$

Open problems

- [BMMN11] showed that Krivine's technique is *not* optimal for rounding SDP solutions to integer solutions
- *Can their rounding scheme be extended to higher ranks?*

- Krivine+ϑ-type rounding and [AMMN06] rounding are favorable for small/large chromatic number resp.
- *Is there some hybrid scheme of these two?*

arXiv:1011.1754 [math.OC]