Classification of topologically protected gates for local stabilizer codes

Robert König

joint work with

Sergey Bravyi
T-W  JCQC
T-W JCQC

Quantum
Tsinghua-Waterloo Joint Center for Quantum Computing
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Local (topological) stabilizer codes

\( D \)-dimensional array of qubits of size \( L \)

*local* stabilizer generators: support of any generator has diameter \( \xi = O(1) \)

code distance \( d \gg \xi \)

\# of encoded qubits \( k \)

examples:

- toric code/surface codes [Kitaev’97, Bravyi, Kitaev’98]
- color codes [Bombin, Martin-Delgado’06]
- 3D self-correcting memories [Haah 12] and [Michnickis 12]
- surface code with twists [Bombin’10]
- ...

\[ Z \quad X \quad X \quad Z \]
\[ \xi \]
Protected gates?

**logical gate**: unitary $U$ preserving codespace $\mathcal{L}$: $U\mathcal{L} = \mathcal{L}$

fault-tolerance properties depend structure of $U$
Protected gates?

**logical gate**: unitary $U$ preserving codespace $\mathcal{L}$: $U \mathcal{L} = \mathcal{L}$

fault-tolerance properties depend on the structure of $U$

example of a protected gate: transversal gate
Protected gates?

**logical gate:** unitary $U$ preserving codespace $\mathcal{L}$: $U\mathcal{L} = \mathcal{L}$

- error locations

fault-tolerance properties depend structure of $U$

example of a protected gate: transversal gate

- preexisting errors do not spread
- faulty unitaries only introduce local errors

*when applying a transversal gate.*
Limitations on transversal encoded gates

**General (non-stabilizer) codes:**

Theorem: Transversal encoded gates generate a **finite group**.

[Eastin, Knill '09]

Proof uses theory of Lie groups.

**2D surface codes:**

Theorem: Suppose the stabilizer group has no generators of weight 2. Then all transversal gates are in the **Clifford group**.

[Sarvepalli, Raussendorf '09]

Proof uses theory of matroids.
A more general notion of protected gates?

- Preexisting errors do not spread.
- Faulty unitaries only introduce local errors when applying a transversal gate.

\textit{depth-1 quantum circuit}
A definition of protected gates

- preexisting errors only spread to a constant-width causal cone
- faulty unitaries introduce errors restricted to causal cone

when applying a gate realized by a constant-depth circuit
A definition of protected gates

• preexisting errors only spread to a constant-width causal cone

• faulty unitaries introduce errors restricted to causal cone

when applying a gate realized by a constant-depth circuit
A definition of protected gates

- preexisting errors only spread to a constant-width causal cone
- faulty unitaries introduce errors restricted to causal cone

When applying a gate realized by a constant-depth quantum circuit

protected gate

equiv

implementable by

constant-depth quantum circuit

• error locations
The Clifford hierarchy and local stabilizer codes

Level 1: Pauli group

Level 2: Clifford group

Level 3: $\pi/8$-gate, Toffoli gate, $\Lambda(S)$, etc.

Level $j + 1$: $C_{j+1} = \{ \bar{U} \in U(2^k) \mid \bar{U}C_1\bar{U}^\dagger \subseteq C_j \}$

[Gottesman, Chuang ’99]
The Clifford hierarchy and local stabilizer codes

Level 1: Pauli group

Level 2: Clifford group

Level 3: $\pi/8$-gate, Toffoli gate, $\Lambda(S)$, etc.

Level $j + 1$: 

$$C_{j+1} = \{ \overline{U} \in U(2^k) \mid \overline{U} C_1 \overline{U}^\dagger \subseteq C_j \}$$

Pauli group $\xrightarrow{\text{Level } j}$ Level $j$

Properties:

• $C_1 \subset C_2 \subset \cdots \subset C_j \subset C_{j+1} \subset \cdots$

• 

$$
\begin{pmatrix}
1 & 0 \\
0 & e^{2\pi i/2^j}
\end{pmatrix} \in C_j \setminus C_{j-1}
$$
The Clifford hierarchy and local stabilizer codes

Level 1: Pauli group

Level 2: Clifford group

Level 3: $\pi/8$-gate, Toffoli gate, $\Lambda(S)$, etc.

Level $j + 1$: $C_{j+1} = \{ \bar{U} \in \text{U}(2^k) \mid \bar{U}C_1\bar{U}^\dagger \subseteq C_j \}$

Theorem: For a $D$-dimensional local stabilizer code: $(D \geq 2)$
encoded gates implementable by a constant-depth circuit belong to the level $D$ of the Clifford hierarchy.
Proof tool I: the union lemma

Def: \( \mathcal{R} \) correctable region \( \iff \) any logical Pauli operator supported on \( \mathcal{R} \) acts as identity on code space

Example: number of qubits \( |\mathcal{R}| < d \)
Proof tool I: the union lemma

Def: $\mathcal{R}$ correctable region :⇔ any logical Pauli operator supported on $\mathcal{R}$ acts as identity on code space

Union lemma:

$\mathcal{R}_1, \mathcal{R}_2$ correctable regions,

distance($\mathcal{R}_1, \mathcal{R}_2$) $> \xi$ $\Rightarrow$ $\mathcal{R}_1 \cup \mathcal{R}_2$ correctable

[Bravyi, Poulin, Terhal '10]
[Haah, Preskill'10]

=diameter of stabilizers
Application of union Lemma: partition of lattice

in $D = 2$: 3 disjoint correctable regions $A, B, C$

by application of the union Lemma

(in $D$: $D + 1$ disjoint correctable regions)
Proof tool II: the cleaning lemma

Def: \( \mathcal{R} \) correctable region \( \iff \) any logical Pauli operator supported on \( \mathcal{R} \) acts as identity on code space

Cleaning lemma: \[ \exists \] stabilizer \( S \) such that \( PS \) is supported outside \( \mathcal{R} \)

[Bravyi, Terhal '08] support of \( S \): contained in \( \xi \)-neighborhood of \( \mathcal{R} \)
Application of cleaning lemma

\[ P = \]

\[ Q = \]

(arbitrary) logical Pauli operators after cleaning
**Application of cleaning lemma**

\[
P = (\text{arbitrary}) \text{ logical Pauli operators after cleaning}
\]

\[
Q = (\text{arbitrary}) \text{ transversal gate } U
\]

\[
QU = UQU^\dagger = (QU \text{ is also transversal!})
\]

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Application of cleaning lemma

\[ P = (\text{arbitrary logical Pauli operators after cleaning}) \]

\[ Q = \]

Application of a transversal gate \( U \)

(constant-depth: similar)

\( U \{ \quad \}

\( Q \{ \quad \equiv \quad \}

\( U^\dagger \{ \quad \} \equiv UQUU^\dagger \)
Transversal gate $U$ & support of ‘group commutator’

$P = \begin{bmatrix} A & B & C & A \\ B & C & A & B & C \\ A & B & C & A \\ B & C & A & B & C \end{bmatrix}$

Pauli

$QU = \frac{UQU^\dagger}{UQU^\dagger} = \begin{bmatrix} A & B & C & A \\ B & C & A & B & C \\ A & B & C & A \\ B & C & A & B & C \end{bmatrix}$

transversal

Claim: $U|_L$ is an encoded Clifford group element
Transversal gate $U$ & support of ‘group commutator’

$P = \begin{array}{cccc}
A & B & C & A \\
B & C & A & B & C \\
A & B & C & A \\
B & C & A & B & C \\
\end{array}$

Pauli

$QU = \begin{array}{cccc}
A & B & C & A \\
B & C & A & B & C \\
A & B & C & A \\
B & C & A & B & C \\
\end{array}$

transversal

$PQUP^\dagger = \begin{array}{cccc}
A & B & C & A \\
B & C & A & B & C \\
A & B & C & A \\
B & C & A & B & C \\
\end{array}$

Claim: $U |_L$ is an encoded Clifford group element
Transversal gate $U$ & support of ‘group commutator’

$P = \begin{bmatrix} A & B & C & A \\ B & C & A & B & C \\ A & B & C & A \\ B & C & A & B & C \end{bmatrix}$

Pauli

$QU = \begin{bmatrix} A & B & C & A \\ B & C & A & B & C \\ A & B & C & A \\ B & C & A & B & C \end{bmatrix}$

transversal

supported on correctable region $A$

$PQU^\dagger Q^\dagger_U = \begin{bmatrix} A & B & C & A \\ B & C & A & B & C \\ A & B & C & A \\ B & C & A & B & C \end{bmatrix}$

Claim: $U\big|_L$ is an encoded Clifford group element
Transversal gate $U$ & support of ‘group commutator’

$P = \begin{bmatrix} A & B & C & A \\ B & C & A & B & C \\ A & B & C & A \\ B & C & A & B & C \end{bmatrix}$

Pauli

$Q_U = \begin{bmatrix} A & B & C & A \\ B & C & A & B & C \\ A & B & C & A \\ B & C & A & B & C \end{bmatrix}$

transversal

$UQUU^\dagger = \begin{bmatrix} A & B & C & A \\ B & C & A & B & C \\ A & B & C & A \\ B & C & A & B & C \end{bmatrix}$

supported on correctable region $A$

$PQUU^\dagger Q_U^\dagger = \begin{bmatrix} A & B & C & A \\ B & C & A & B & C \\ A & B & C & A \\ B & C & A & B & C \end{bmatrix}$

$\Rightarrow PQUU^\dagger Q_U^\dagger \mid_{\mathcal{L}} \propto I_{\mathcal{L}}$

by definition of correctable regions

Claim: $U \mid_{\mathcal{L}}$ is an encoded Clifford group element
Transversal gate $U$ & support of ‘group commutator’

$P = \begin{bmatrix} A & B & C & A \\ B & C & A & B & C \\ A & B & C & A \\ B & C & A & B & C \end{bmatrix}$

Pauli

$Q_U = \begin{bmatrix} A & B & C & A \\ B & C & A & B & C \\ A & B & C & A \\ B & C & A & B & C \end{bmatrix}$

transversal

$PQ_UP^\dagger Q_U^\dagger = \begin{bmatrix} A & B & C & A \\ B & C & A & B & C \\ A & B & C & A \\ B & C & A & B & C \end{bmatrix}$

supported on correctable region $A$

$\Rightarrow PQ_UP^\dagger Q_U^\dagger|_L \propto I_L$ by definition of correctable regions

$\Rightarrow Q_UP|_L = \pm PQ_U|_L$ for all logical Pauli op $P, Q$

$\Rightarrow \textbf{Claim: } U|_L$ is an encoded Clifford group element
Generalizing to higher dimensions

in $D = 2$

3 disjoint correctable regions $A, B, C$
Generalizing to higher dimensions

in $D = 2$

3 disjoint correctable regions $A, B, C$
Generalizing to higher dimensions

In $D = 2$
3 disjoint correctable regions $A, B, C$

In $D = 3$
4 disjoint correctable regions $A, B, C, D$

In $D$: $D + 1$ disjoint correctable regions
The Clifford hierarchy and local stabilizer codes

Level $j + 1$: $\mathcal{C}_{j+1} = \{ \bar{U} \in U(2^k) \mid \bar{U} \mathcal{C}_1 \bar{U}^\dagger \subseteq \mathcal{C}_j \}$

Pauli group

Level $j$

Theorem: For a $D$-dimensional local stabilizer code: $(D \geq 2)$ protected gates belong to $\mathcal{C}_D$.

application of a constant-depth circuit $U$

Clifford group $\mathcal{C}_2$

$\mathcal{C}_3$
Theorem: For a $D$-dimensional local stabilizer code: $\quad (D \geq 2)$
protected gates belong to $\mathcal{C}_D$.

(Code deformation version) sequence of codes $\mathcal{L}^{(1)}, \ldots, \mathcal{L}^{(t)}$

$\mathcal{L}^{(1)} \rightarrow \mathcal{L}^{(2)} \rightarrow \cdots \rightarrow \mathcal{L}^{(t)}$

constant-depth circuit
constant-depth circuit
constant-depth circuit

overall logical operation belongs to $\mathcal{C}_D$
Consequences for universality?

Level $j + 1$:

$$C_{j+1} = \{ \bar{U} \in U(2^k) \mid \bar{U}C_1\bar{U}^+ \subseteq C_j \}$$

Pauli group \quad Level $j$

Theorem: For a $D$-dimensional local stabilizer code:

protected gates belong to $C_D$. ($D \geq 2$)

Corollary:

2-dimensional local stabilizer code

Corollary:

$\{L_L\}_L$ family of $D$-dimensional local stabilizer codes such that $k = k(L)$ independent of $L$
Proof of Corollary

Claim:

\( \{ \mathcal{L}_L \} \) family of \( D \)-dimensional local stabilizer codes such that

\( k = k(L) \) independent of \( L \)

\( h = \text{const.} \)

set of gates \( \mathcal{P}_h \) implementable by depth-\( h \) circuit

generates group

\[ \langle \mathcal{P}_h \rangle \subset \mathcal{C}_D \]

finite
Proof of Corollary

Claim:
\{L_L\}_L family of $D$-dimensional local stabilizer codes such that
\[ k = k(L) \] independent of $L$

Proof by contradiction:

(i) Suppose $\exists U_1, \ldots, U_m \in \mathcal{P}_h$ such that
\[ U = U_1 U_2 \cdots U_m \notin \mathcal{C}_D \]
Proof of Corollary

Claim:

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\[
U = U_1 U_2 \cdots U_m \notin \mathcal{C}_D
\]

(ii) wlog: \( m = m(k) \) is constant independent of \( L \)

set of gates \( \mathcal{P}_h \) implementable by depth-\( h \) circuit generates group \( \langle \mathcal{P}_h \rangle \subset \mathcal{C}_D \) finite
Proof of Corollary \( h = \text{const.} \)

Claim:
\( \{\mathcal{L}_L\}_L \) family of \( D \)-dimensional local stabilizer codes such that \( k = k(L) \) independent of \( L \)

Proof by contradiction:
(i) Suppose \( \exists U_1, \ldots, U_m \in \mathcal{P}_h \) such that
\[
U = U_1 U_2 \cdots U_m \not\in \mathcal{C}_D
\]
(ii) wlog: \( m = m(k) \) is constant independent of \( L \)
(iii) \( U \) implementable by depth-\( (m \cdot h) \) circuit \( \xRightarrow{\text{Thm}} U \in \mathcal{C}_D \)

set of gates \( \mathcal{P}_h \) implementable by depth-\( h \) circuit generates group \( \langle \mathcal{P}_h \rangle \subset \mathcal{C}_D \) finite
Proof of Corollary

Claim:
\{L_L\}_L family of \(D\)-dimensional local stabilizer codes such that 
\(k = k(L)\) independent of \(L\)

Proof by contradiction:

(i) Suppose \(\exists U_1, \ldots, U_m \in \mathcal{P}_h\) such that
\[U = U_1 U_2 \cdots U_m \not\in \mathcal{C}_D\]

(ii) wlog: \(m = m(k)\) is constant independent of \(L\)

(iii) \(U\) implementable by depth-\((m \cdot h)\) circuit \(\Rightarrow U \in \mathcal{C}_D\)
Alternatives to getting universality?

- 3D stabilizer codes with universal gate sets \( k = k(L) \)
  

- magic state distillation
  
  Raussendorf, Harrington, Goyal NJP 9, 199 (2007)

- non-stabilizer codes
  
Thank you for your attention!

arXiv:1206.1609