Rank-one and Quantum XOR Games

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Two Player, One Round Games

Alice

Referee

Bob

\[ V(a, b, x, y) \in \{0, 1\} \]

- Computational Complexity
  - Interactive proof systems
  - Efficient proof verification
  - PCP theorem
  - Hardness of approximation
Two Player, One Round Games

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- Computational Complexity
  - Interactive proof systems
  - Efficient proof verification
  - PCP theorem
  - Hardness of approximation
- Nonlocality/Bell inequalities
Classical XOR Games

Classical XOR games: \( \{a, b\} \in \{0, 1\} \)

\[ V(a, b, x, y) = V(a \oplus b, x, y). \]
Biases of (Classical) XOR Games

- Bias = $2 \times$ Maximum Success Probability $- 1$
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- Bias = 2 × Maximum Success Probability − 1
- Unentangled bias $\omega(G)$ (with classical resources)
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  (Players can share maximally entangled state of arbitrary dimension)
- Entangled bias $\omega^*(G)$
Classical XOR Games versus Quantum XOR

- For all classical XOR games $G$, we have
  $$\omega(G) \leq \omega^*(G) \leq K \omega(G),$$
  where $1.67 \leq K \leq 1.783$, [CHTW04].

Quantum XOR: Unbounded advantage provided by entanglement

Classical XOR: Maximally entangled states are optimal resource. [CHTW04]

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  $$\omega^*(G^\otimes n) = (\omega^*(G))^n$$
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- For Quantum XOR games $G$, $\omega^*(G)$ can be approximated up to a constant multiplicative factor using SDP. ✓
- [CSUU08] Classical XOR games satisfy *Perfect Parallel Repetition*: ✓
  \[ \omega^*(G \otimes^n) = \omega^*(G)^n \]
- Quantum XOR: Unbounded Violation of Perfect Parallel Repetition
  \[ \omega^*(C_n \otimes^2) \geq \frac{n}{2} \omega^*(C_n)^2 \]
Quantum XOR Games

Referee prepares (known) state  $|\psi_i\rangle_{AB} |i\rangle_R \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_R$ and sends register $A$ to Alice, $B$ to Bob.
Referee has private register $\mathcal{H}_R$. 

$\sum_i p_i^{1/2} |\psi_i\rangle_{AB} |i\rangle_R \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_R$
Quantum XOR Games

Referee prepares (known) state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_R$ and sends register $A$ to Alice, $B$ to Bob.
Referee has private register $\mathcal{H}_R$. 
Quantum XOR Games

Alice and Bob share an entangled state $|\xi\rangle \in \mathcal{H}_{A'} \otimes \mathcal{H}_{B'}$. Alice and Bob apply ±1-observables $X_{AA'} = X^0 - X^1$, $Y_{BB'} = Y^0 - Y^1$. Return outcomes $a, b \in \{0, 1\}$ to Referee.
Quantum XOR Games

If $a \oplus b = 0$, \( \{ \Pi_0^{ACC}, \text{id}_R - \Pi_0^{ACC} \} \)

If $a \oplus b = 1$, \( \{ \Pi_1^{ACC}, \text{id}_R - \Pi_1^{ACC} \} \)

Referee measures private register, depending on parity of Alice and Bob’s responses.
Example: $T_n$

Let $|\psi_n\rangle$ be the maximally entangled state in $n$ dimensions.

$T_n$: Alice and Bob sent one of

$$|\phi_0\rangle = \frac{1}{\sqrt{2}} (|0\rangle|0\rangle + |\psi_n\rangle)$$

$$|\phi_1\rangle = \frac{1}{\sqrt{2}} (|0\rangle|0\rangle - |\psi_n\rangle)$$

with equal probability.

If $|\phi_0\rangle$, respond with answers of even parity.

If $|\phi_1\rangle$, respond with answers of odd parity.

Orthogonal ✓  Locally distinguishable ?
Unbounded advantage of $\omega^*(G)$ over $\omega^{me}(G)$ and $\omega(G)$

\[ \omega(T_n) = \omega^{me}(T_n) = \frac{1}{\sqrt{n}} \]

\[ \omega^*(T_n) = 1 \]
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\[
\omega^*(T_n) = 1
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$\omega^*(T_n) = 1$ can only be achieved in limit of infinite entanglement.
($T_2 \leftrightarrow$ LTW’s coherent state exchange game.)
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Classical XOR games: maximally entangled states are optimal resource.

Entanglement provides advantage of at most small constant multiplicative factor: Grothendieck/Tsirelson.
Referee chooses $k \in \{1, \ldots, n\}$ randomly.
Sends one of the two states

$$|\phi_{0k}\rangle = \frac{1}{\sqrt{2}} (|0\rangle|k\rangle + |k\rangle|0\rangle)$$

$$|\phi_{1k}\rangle = \frac{1}{\sqrt{2}} (|0\rangle|k\rangle - |k\rangle|0\rangle)$$

each chosen with probability $\frac{1}{2}$ to Alice and Bob.

If $|\phi_{0k}\rangle$, respond with answers of even parity.
If $|\phi_{1k}\rangle$, respond with answers of odd parity.
Suppose Alice and Bob play two games simultaneously and must win both “sub-games” in order to win.

For classical XOR games, we have

\[
\omega^*(G \otimes G) = \omega^*(G)^2.
\]
Large Violation of Perfect Parallel Repetition

- Suppose Alice and Bob play two games simultaneously and must win both “sub-games” in order to win.

For classical XOR games, we have

$$\omega^*(G \otimes G) = \omega^*(G)^2.$$  

However for Rank-one Quantum & Quantum XOR games:

$$\omega^*(C_n) = \frac{1}{n}$$

$$\omega^*(C_n \otimes C_n) \geq \frac{1}{2n} \gg (\omega^*(C_n))^2$$
Theorem

There exists a polynomial-time algorithm which, given as input an explicit description of a quantum XOR game $G$, outputs two numbers $\omega^{nc}(G)$ and $\omega^{os}(G)$ such that

$$\omega(G) \leq \omega^{me}(G) \leq \omega^{nc}(G) \leq 2\sqrt{2}\omega(G),$$

$$\omega^{*}(G) \leq \omega^{os}(G) \leq 2\omega^{*}(G).$$
Techniques

Theorem (Grothendieck’s Inequality)

Suppose that $s_i$ and $t_j$ are real numbers such that $|s_i|, |t_j| \leq 1$.

Suppose that $a_{ij}$ are real numbers such that $\left| \sum_{i,j} a_{ij} s_i t_j \right| \leq 1$. Then

$$\left| \sum_{ij} a_{ij} \langle \xi_i | \eta_j \rangle \right| \leq k,$$

for all vectors $\xi_i, \eta_j$ in the unit ball of a real Hilbert space $\mathcal{H}$. It is known that $1.67 \leq k \leq 1.782$.

From this it follows that for a classical XOR game,

$$\omega^*(G) \leq k \omega(G).$$
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Noncommutative and Operator-space extensions of Grothendieck’s inequality allow us to relate biases of Quantum XOR games to SDP’s.
Quantum Games

Referee prepares (known) state \( |\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_R \) and sends register \( A \) to Alice, \( B \) to Bob.
Alice and Bob share an entangled state $|\xi\rangle \in \mathcal{H}_{A'} \otimes \mathcal{H}_{B'}$. Alice and Bob apply arbitrary local unitaries $U_{AA'}, V_{BB'}$ and then send registers $A$ and $B$ back to referee.
Quantum Games

Referee performs measurement with projective measurements:

\[ \{ P_{\text{ACCEPT}}, P_{\text{REJECT}} = \text{Id} - P_{\text{ACCEPT}} \} \]

\[ (U_{AA'} \otimes V_{BB'} \otimes \text{id}_R) |\psi\rangle |\xi\rangle \]
Referee performs measurement with projective measurements:

\[ \{ P_{\text{ACCEPT}} = |\gamma\rangle\langle\gamma|, P_{\text{REJECT}} = \text{Id} - P_{\text{ACCEPT}} \} \]

Maximum Success Probability = \( \omega_1^*(G) \)
Rank-one Quantum Games $\leftrightarrow$ Quantum XOR Games

To each Quantum XOR Game $G$, one can associate a Rank-one Quantum Game $G'$ such that

$$ (\omega^*(G))^2 = \omega^*_1(G) $$

To each Rank-one Quantum Game $G'$, one can associate a Quantum XOR Game $G''$ such that

$$ (\omega^*(G''))^2 = \omega^*_1(G') $$

Thus the previous results about SDP’s and Parallel repetition can be phrased in terms of either Rank-one Quantum Games or Quantum XOR games.
Summary 1

- Classical: $\omega^*(G) \leq K\omega(G)$ ✓
- Quantum: Unbounded advantage $\omega^*(T_n) = \sqrt{n}\omega(T_n)$ ✗
- Classical: Maximally entangled state is optimal resource. ✓
- Quantum: Unbounded advantage $\omega^*(T_n) = \sqrt{n}\omega^{me}(T_n)$ ✗
- Classical: $\omega^*(G)$ can be computed using SDP ✓
- Quantum: $\omega^*(G)$ can be approximated up to constant factor using SDP ✓
- Classical: Satisfies Perfect Parallel Repetition: ✓

$$\omega^*(G^\otimes 2) = \omega^*(G)^2$$

Quantum: Unbounded Violation of Perfect Parallel Repetition

$$\omega^*(C_n^\otimes 2) \geq \frac{n}{2}\omega^*(C_n)^2$$
Summary 2

- Generalization of classical XOR games using quantum messages.
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Application of deep generalizations of Grothendieck’s Inequality to problems in quantum information theory.
Summary 2

- Generalization of classical XOR games using quantum messages.
- Rich class of games that displays properties of entanglement not seen in classical case.
- Remain tractable with efficient approximation algorithms for biases.
- Application of deep generalizations of Grothendieck’s Inequality to problems in quantum information theory.
- Operator space theory provides both examples and techniques for studying these quantum games.
Thank You!

- Rank-one Quantum Games, arXiv: 1112.3563
- Quantum XOR Games, arXiv: 1207.4939

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