Negative Quasi-Probability as a Resource for Quantum Computation

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Big Picture Question

What resources are necessary and sufficient for quantum computational speedup?
Resources for Quantum Computation?

Some Candidates

- Entanglement?
- Purity?
- Coherence?
- Discord? (probably not)

Quantum Resources

Resources arise from operational restrictions on the quantum formalism.
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Resources for Fault Tolerance

Goal
The goal is to characterize resources for fault tolerant quantum computation.

Fault Tolerance
- Stabilizer operations are a typical fault tolerant set.
- This defines a natural restriction on the set of quantum operations.
- This set is efficiently simulatable by the Gottesman-Knill protocol.
- Thus we need injection of resource states.
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Magic State Computing (Bravyi, Kitaev 2005)

Magic State Model

- Operational restriction: perfect stabilizer operations (states, gates and projective measurement)
- Additional resource: preparation of non-stabilizer state $\rho_R$

Magic State Distillation

- Consume many resource states $\rho_R$ to produce a few very pure resource states $\tilde{\rho}_R$
- Inject $\tilde{\rho}_R$ to perform non-stabilizer unitary gates (using only fault tolerant stabilizer operations)

A Sharper Question

Do all non-stabilizer states promote stabilizer computation to quantum computation?
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Main Result

There is a large class of non-stabilizer quantum states (*bound magic states*) that are not useful for magic state distillation.

Quantum circuits composed of stabilizer operations composed of stabilizer operations and bound magic states are efficiently classically simulatable. This is an extension of Gottesman-Knill to non-stabilizer inputs.
Quasi-Prob. Representation

A linear map from Hermitian operators to real numbers, $W : L(\mathcal{H}_d^n) \to \mathbb{R}^{d^{2n}}$. In particular:
- quantum states $\to$ quasi-probability distributions
- POVM elements $\to$ conditional quasi-probability distributions.

Negativity

Some states/measurements must be *negatively represented*. (Emerson, Ferrie 2009).

Subtheory

Choice of positively represented subtheory is largely arbitrary.
Insight
Choice of quasi-probability representation can reflect operational restriction.

Wigner Representation
Stabilizer operations have positive representation. (Gross 2006)

Negative Probabilities
Ancilla preparation may be negatively represented.

Figure: Wigner representation of qutrit $|0\rangle$ state

Figure: Wigner representation of qutrit $|0\rangle - |1\rangle$ state
Observation
Negative Wigner representation is a resource that can not be created by stabilizer operations.

Proof
Let $\rho \in L(\mathbb{C}_{d^n})$ be an $n$ qudit quantum state with positive Wigner representation. Observe the following:

1. $U\rho U^\dagger$ is positively represented for any Clifford (stabilizer) unitary $U$.
2. $\rho \otimes S$ is positively represented for any stabilizer state $S$.
3. $M\rho M/\text{Tr}(M\rho M)$ is positively represented for any stabilizer projector $M$. 
Discrete Hudson’s Theorem (Gross 2006)
Pure states have positive representation if and only if they are stabilizer states.

Positive Representation ≡ Stabilizer State?
Do all non-stabilizer states have negative Wigner representation?
Stabilizer Polytope

- Define convex polytope with stabilizer states as its vertices
- Can be equivalently defined by set of halfspaces - “facets”

Non-Negativity Specifies Facets

The Wigner simplex has $d^2$ facets, shared with the stabilizer polytope
Figure: Slice defined by fixing six entries of the Wigner function and varying the remaining through their possible values to create the plot.
Extended Gottesman-Knill Theorem

**Scope**
- Prepare $\rho$ with positive representation
- Act on input with Clifford $U_F$ (corresponding to linear size $F$)
- Perform measurement $\{E_k\}$ with positive representation

**Simulation Protocol**
- Sample phase space point $(u, v)$ according to distribution $W_\rho(u, v)$
- Evolve phase space point according to $(u, v) \rightarrow F^{-1}(u, v)$
- Sample from measurement outcome according to $\tilde{W}_{\{E_k\}}(u, v)$

See also *Positive Wigner functions render classical simulation of quantum computation efficient*, A. Mari and J. Eisert
**Linear Optics**

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**Table**: Comparison of Odd and Infinite Dimensional Formalisms

**Results**

- There exist mixed states with positive Wigner representation that are not convex combinations of gaussian states (Bröcker and Werner 1995)
- Computations using linear optical transformations and measurements as well as preparations with positive Wigner function can be efficiently classically simulated.$^a$

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$^a$Veitch, Wiebe, Ferrie and Emerson (2012)
Summary

- Negative Wigner representation resource for stabilizer restriction
- Extended Gottesman-Knill
- Bound states for magic state distillation

Future Work

- Does this extend to other operational restrictions?
- Is negativity sufficient for distillability?
- Resource theory for stabilizer formalism?

Paper Reference

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