An area law and sub-exponential algorithm for 1D systems.

A Basic Question

Lesson from Quantum Complexity Theory:

Finding ground/low energy states is QMA hard, even for 1D systems. So analysis of many body physics is impossible! And yet . . . condensed matter physicists do it! Heuristic techniques, DMRG, have been very successful for 1D systems. Is there a principled phenomenon behind this? Is there a clean well defined class of quantum many body systems that we can analyze?
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Gap and Area Law

The size of the gap between the lowest and second lowest eigenstate:
- QMA-complete require an inverse polynomial size gap.
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A first point of entry is a remarkable conjecture:

\[
\text{Area Law: } \text{Given a gapped local Hamiltonian, for any subset } S \text{ of particles, the entanglement entropy of } \rho_S \text{, the reduced density matrix of the ground state restricted to } S, \text{ is bounded by the surface area of } S \text{ i.e. the number of local interactions between } S \text{ and } \overline{S}. \]
Basic Questions

- Can you prove an area law?
- If so, do these states have small working descriptions?
- Can they be efficiently computed?
Concretely in 1D

Given:

- $n$ $d$-dimensional particles on a line, $\mathcal{H} = (\mathbb{C}^d)^\otimes n$,
- local operators $0 \leq H_i \leq 1$ acting non-trivially on the $i$th and $i+1$st particle.
- a Hamiltonian $H = \sum_i H_i$ with a gap $\epsilon$ between the energy of the ground state and the next lowest energy.
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Goal: structural properties of the ground state $|\Gamma\rangle$. 
Result: 1D Area Law

Previous results for 1D:

- Hastings (2007) with bound $e^{O(\log d/\epsilon)}$.
  - Existence of an MPS with polynomial bond dimension.
  - Finding an approximation to the ground state is $\in NP$.
- Arad, Landau, Vazirani (2011): $\tilde{O}\left(\frac{\log d}{\epsilon}\right)^3$ for frustration free system.
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This result:

**Theorem:** The entanglement entropy of the ground state of a 1D gapped Hamiltonian is bounded by $\tilde{O}\left(\frac{\log^3 d}{\epsilon}\right)$

- Exponential improvement of the bound.
- Bound the cusp of a 2D sub-volume law.
- Implies a sublinear bond dimension MPS which leads to . . .
Sub-exponential algorithm

**Theorem**: There is a subexponential time algorithm for finding an inverse polynomial approximation to ground state of a 1D gapped Hamiltonian.

*Combines sublinear bond dimension with a dynamical programing algorithm (Aharonov, Arad, Irani, 2009).*
Preliminaries: Entanglement Rank

For a vector $v \in \mathcal{H}_1 \otimes \mathcal{H}_2$ with Schmidt decomposition $v = \sum_{i=1}^{D} a_i \otimes b_i$, has entanglement rank $D$.

Operators

Local operator $H_i$ can only increase the entanglement rank across $i, i + 1$ by $d^2$. 
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- Operators of the form \( \sum_{i}^C A_i \otimes B_i \) can only increase the entanglement rank by a factor of \( C \).
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- Local operator $H_i$ can only increase the entanglement rank across $i, i + 1$ by $d^2$. 
Preliminaries: Functional calculus of an operator

What does the operator $f(H) = H^2 - 2H - 1$ look like?

- Same eigenspaces as $H$,
- Eigenvalue $x$ for $H$ becomes eigenvalue $f(x)$ for $f(H)$.
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![Depiction of f(H)](image)

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Proof main idea: moving closer while not increasing entanglement too much

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- It approximately projects onto the ground state:
  $$K = f(H)$$

Such an operator is a $(\mathcal{D}, \Delta)$ Approximate Ground State Projection (AGSP).

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Proof main idea: moving closer while not increasing entanglement too much

We are looking for an operator $K$ with 2 properties:

- It approximately projects onto the ground state:
  \[ K = f(H) \]

- It doesn’t increase the entanglement too much:

Such an operator is a $(\Delta, \Delta)$ Approximate Ground State Projection (AGSP).
The consequence of a good AGSP: An area law

**Theorem (Area Law)** [Arad, Landau, Vazirani] The existence of an AGSP $K$ for which $D \Delta < 1/2$ proves that the ground state has entropy $O(1) \log D$. 
Building good AGSP’s: reduce the norm

Looking for low entanglement operators that look like:

$$f(x) \Delta \frac{\epsilon}{||H||}$$

Smaller $||H||$ would be better but we don’t want to lost the local structure around the cut.

**Solution:** Replace $H = \sum_i H_i$ with $H' = H_L + H_1 + H_2 + \cdots + H_s + H_R$. 

$$H_L \quad \ldots \quad s+1$$
Building good AGSP’s: Chebyshev polynomials

Chebyshev polynomials: small in an interval:

Chebyshev Polynomials

T1 - brown  T2 - black  T3 - red  T4 - pink  T5 - blue  T6 - green
A good AGSP

A dilation and translation of the Chebyshev polynomial gives:

\[ K = C_l(H') \]

with

\[ \Delta = e^{-\ell \sqrt{\epsilon}} \sqrt{||H'||}. \]
Entanglement Increase due to a single term of $(H')^\ell$

$$(H')^\ell = \sum (\text{product of } H_j).$$

For a single term:
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- Across some cut, an average number of terms are involved \(\rightarrow d^{2\ell/s} \).

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- Roundtrip cost of going and coming back from center cut: \(\rightarrow d^s\).
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**Total:** \(d^{2\ell}/s + s\)
Entanglement Increase Analysis of $(H')^\ell$

**Problem:** Too many $(s^\ell)$ terms in naive expansion of $(H')^\ell$.
Entanglement Increase Analysis of \( (H')^\ell \)

**Problem:** Too many \( (s^\ell) \) terms in naive expansion of \( (H')^\ell \).

Need to group terms in a nice way but it all works out with total entanglement increase of the same order as the single term.
Putting things together: Area Law for $H'$

Chebyshev $C_\ell(H')$ has $\Delta \approx e^{-O(\ell\sqrt{\epsilon}/\sqrt{s})}$:

Entanglement analysis yields $D \approx O(s^{\ell/s+s})$.

Choosing $\ell = s^2$ yields $\log(D\Delta) \approx -s^{3/2}\sqrt{\epsilon} + s \log d$. Approximate equality occurs with $s \approx \log^2 d/\epsilon$ which yields $D \approx \log^3 d/\epsilon$. 
From $H'$ to $H$: truncation and the definition of $H_L$ and $H_R$

The truncation, $A \leq^t$ of an operator $A$:


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From $H'$ to $H$: truncation and the definition of $H_L$ and $H_R$

The **truncation**, $A^{\leq t}$ of an operator $A$:

\[ A^{\leq t} = f(A) \]

**Definition of $H_L$ and $H_R$ using truncation:**

\[ H_L = \left( \sum_{i < s+1} H_i \right)^{\leq t} \quad \text{and} \quad H_R = \left( \sum_{i > s+1} H_i \right)^{\leq t} \]
From $H'$ to $H$: robustness of truncation

**Question** How does the Hamiltonian $H' = H_L + H_1 + \cdots + H_s + H_R$ compare to $H = \sum_j H_j$?
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**Robustness Theorem:** The gaps of $H$ and $H'$ are of the same order and the ground states of $H$ and $H'$ are within $\exp(-t)$. 

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Area law for $H$ now follows by starting with a constant truncation level $t = t_0$ and then letting it grow to $t = O(\log n)$. 
Summary

The structural engine for these results are AGSP's.

Area law

Subexponential time algorithm

Gap
Summary

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Schuch, Cirac, Verstraete


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Where to go from here

Towards an area law for 2D... any improvement in the entropy bound $\tilde{O}(\log d/\epsilon)$ would produce a sub-volume law for 2D systems.

Towards better approximation algorithms for 1D... [Landau, Vidick, Vazirani].

Towards more local algorithms in 1D... Of independent interest: entanglement rank has a "random walk" type behavior (added entanglement of $H_\ell$ is $dO(\sqrt{\ell})$).

Of independent interest: robustness theorem of truncation.

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