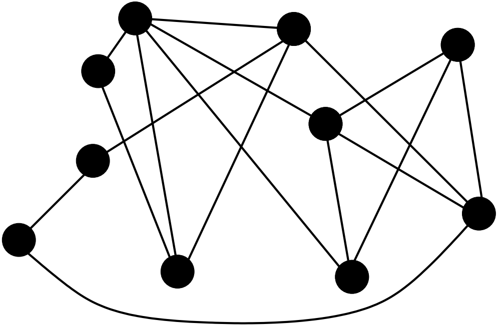
Vertex Sparsification and Oblivious Reductions

Ankur Moitra, MIT

September 14, 2010

The Minimum Bisection Problem

Goal: Minimize cost of bisection



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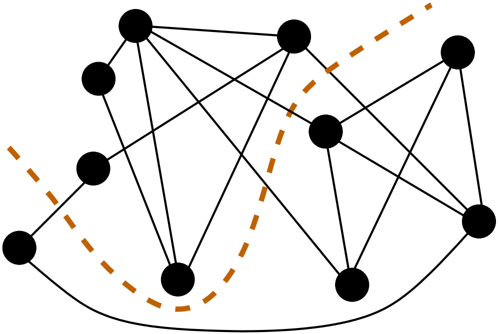

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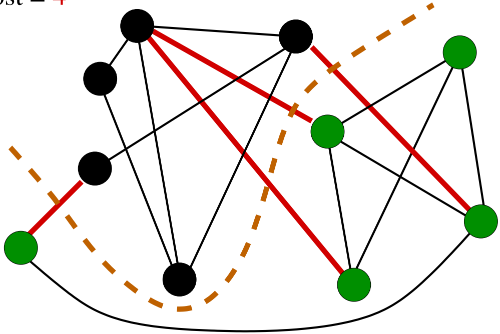
cost = 4



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1. Applications through Divide-and-Conquer: VLSI design, sparse matrix computations, approximation algorithms

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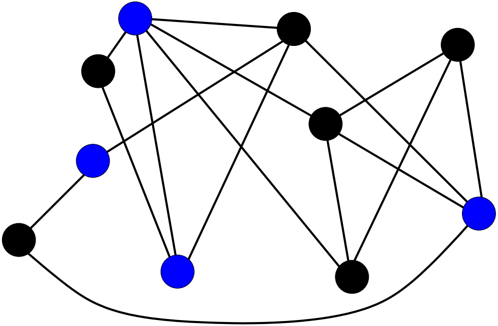
The Steiner Minimum Bisection Problem

Goal: Minimize cost of a bisection of the k blue nodes



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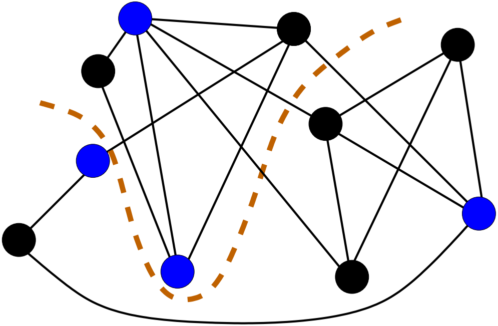

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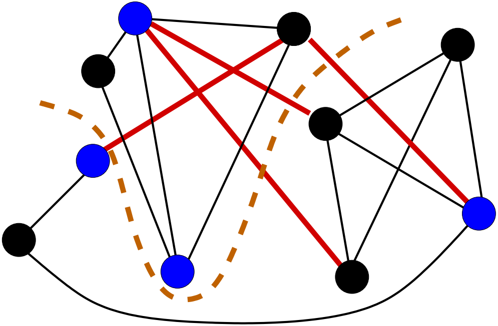

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3. $O(\frac{\log k}{\log \log k})$ 0-**extension** for k terminals [Fakcharoenphol, Harrelson, Rao, Talwar 2003]

A Meta Question

Given

A $\text{poly}(\log n)$ *approximation algorithm (integrality gap or competitive ratio) for an optimization problem characterized by cuts or flows*

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Meta Question

Can we give a poly($\log k$) *approximation algorithm (integrality gap or competitive ratio)?*

Yes we can...

Highlights

1. **Approximation Guarantees Independent of the Graph Size:** We give the first $poly(\log k)$ approximation algorithms (or competitive ratios) for:

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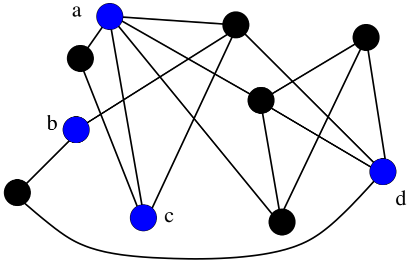
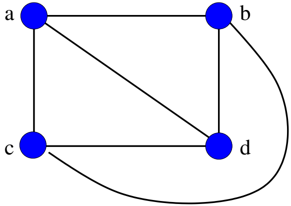
General Approach: Cut Sparsifiers

Graph G=(V,E)



Sparsifier G'=(K,E')



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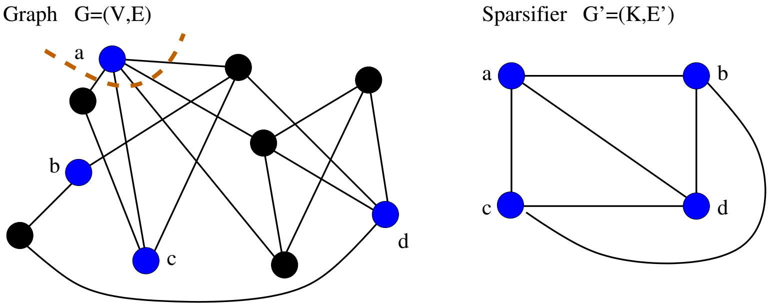

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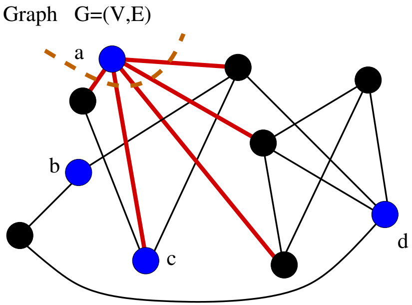

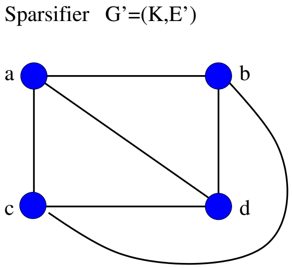

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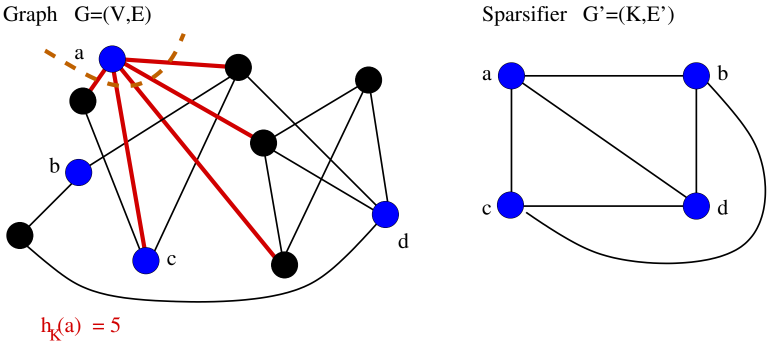

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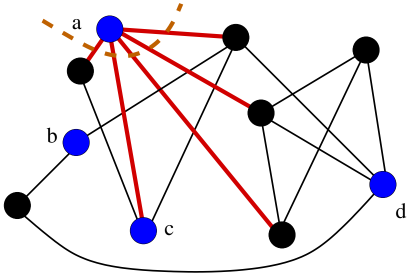

$h_K(a) = 5$

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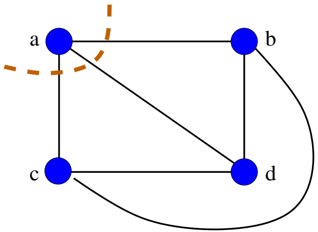

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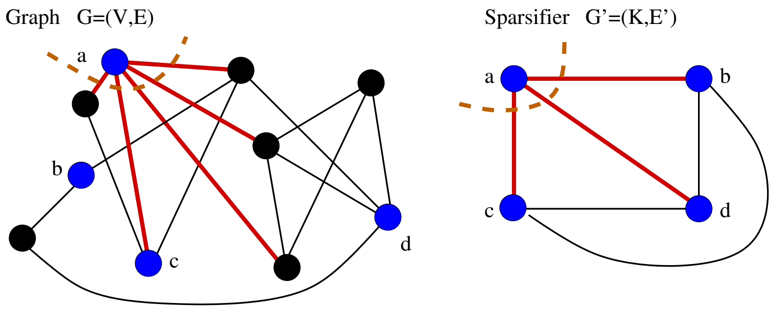

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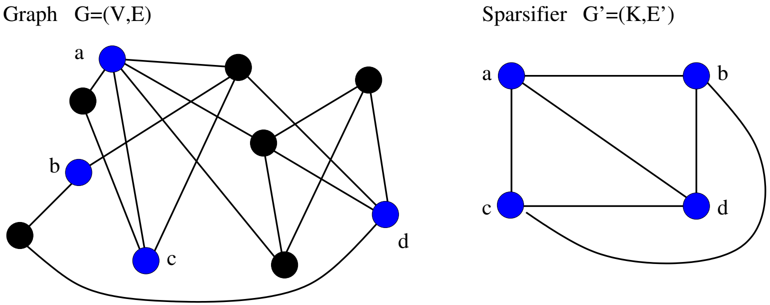

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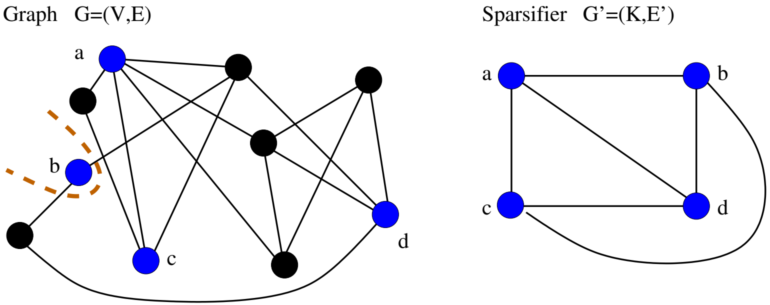

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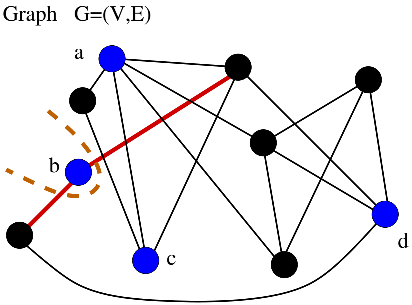

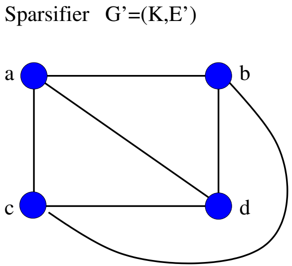

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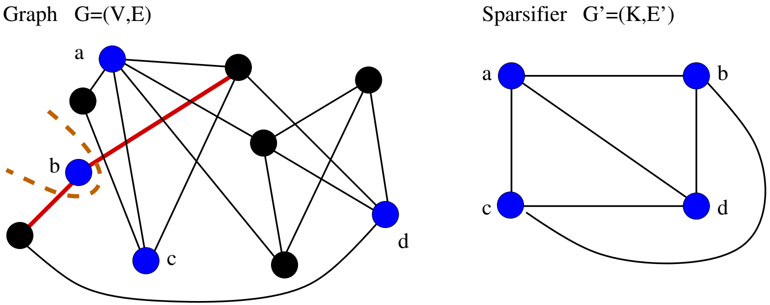
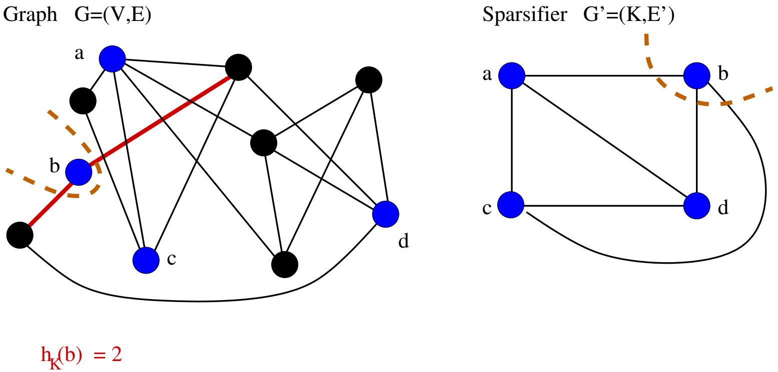


$h_K(b) = 2$

General Approach: Cut Sparsifiers



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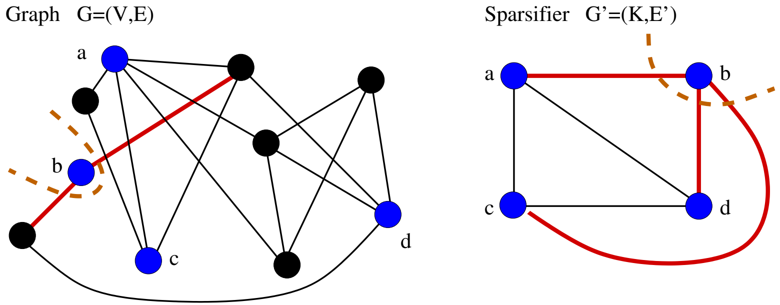

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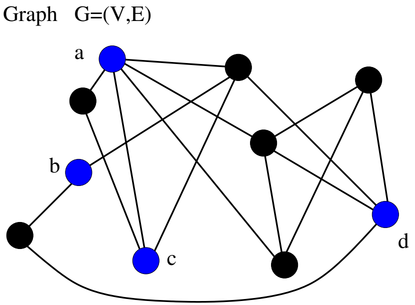

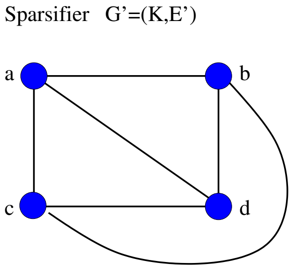

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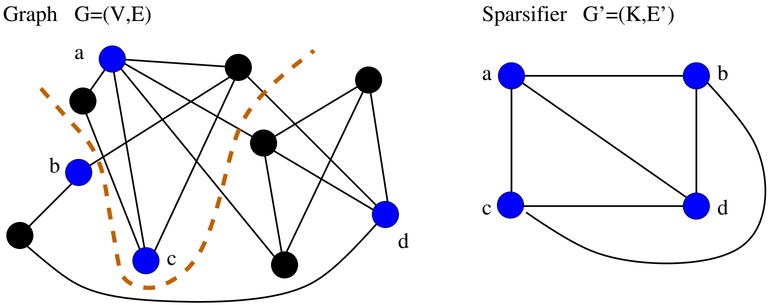

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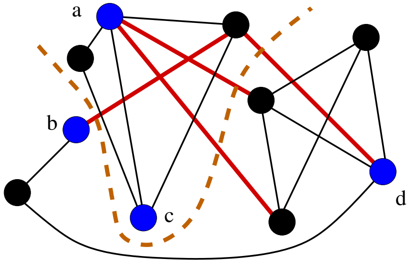

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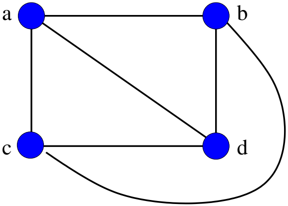

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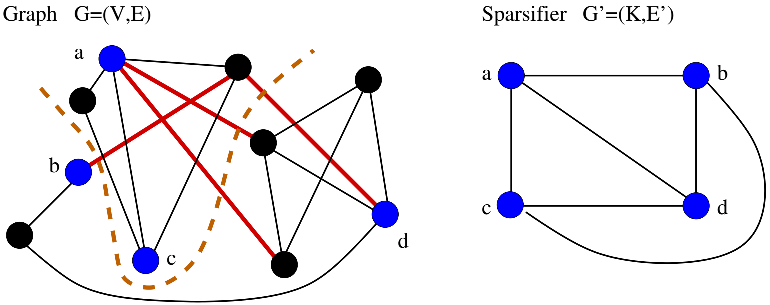

$h_K(ac) = 4$

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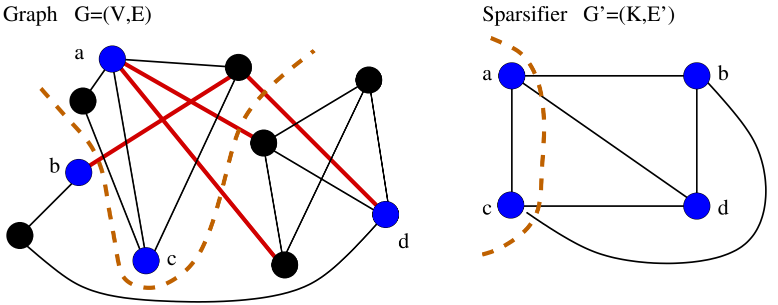

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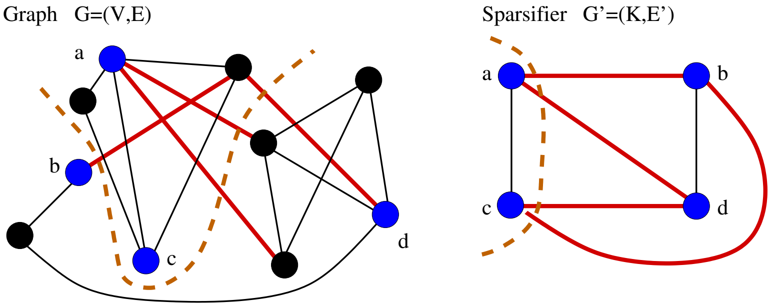

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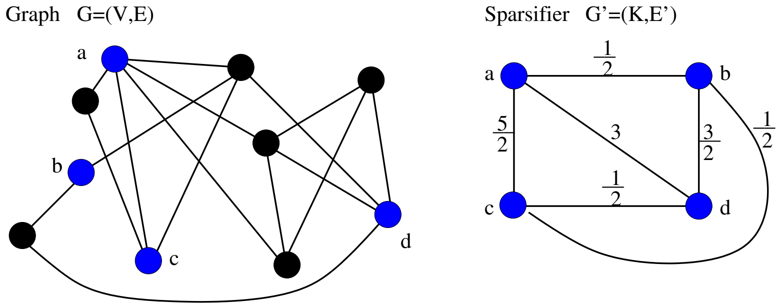

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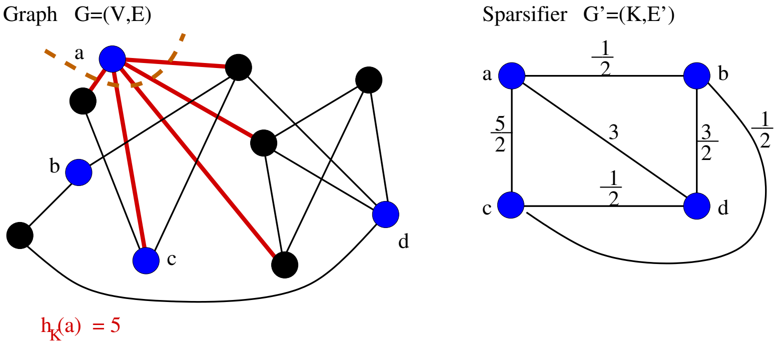

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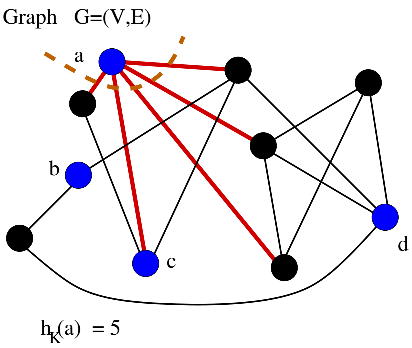

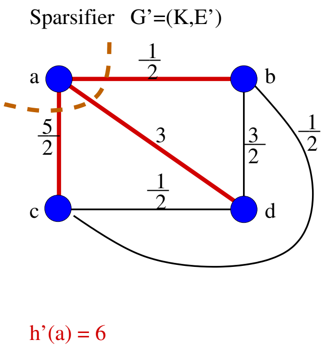

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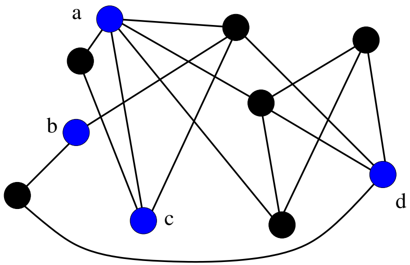

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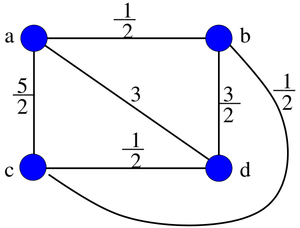
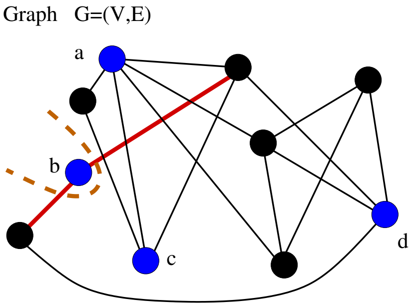


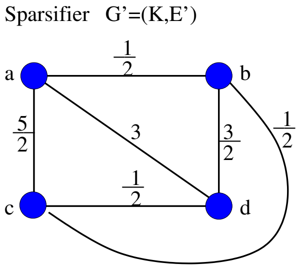
$h'(a) = 6$

General Approach: Cut Sparsifiers



$h_K(a) = 5$

$h_K(b) = 2$



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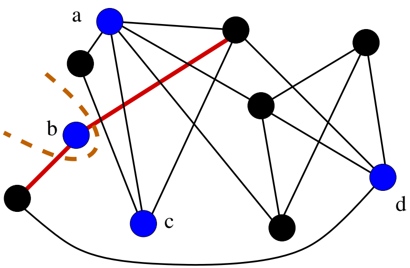

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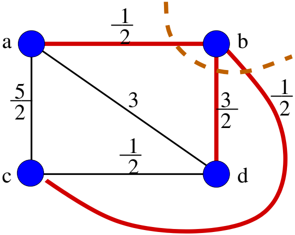
Sparsifier G'=(K,E')



h'(a) = 6
h'(b) = 2.5

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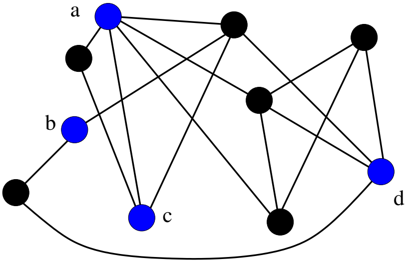

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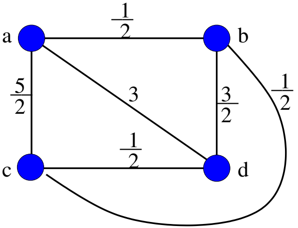

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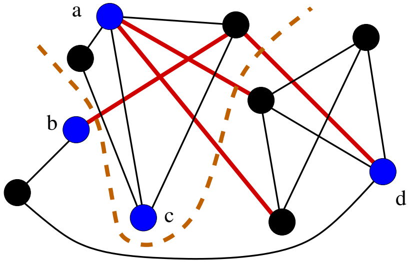

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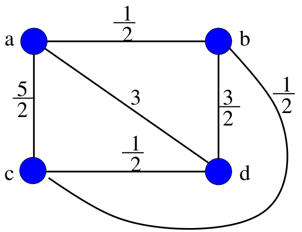
$h_K(ac) = 4$

Sparsifier G'=(K,E')



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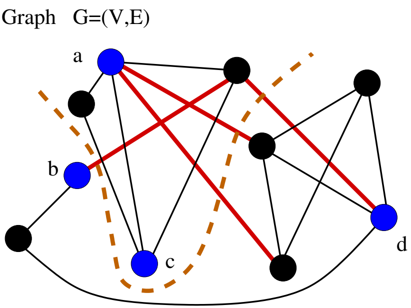

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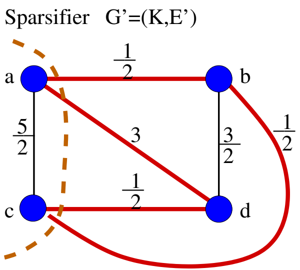

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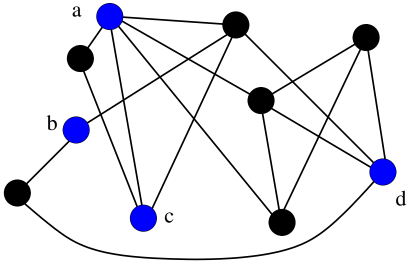
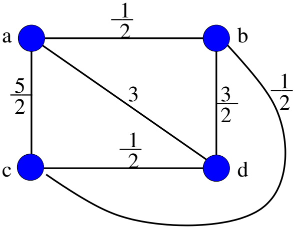


$h_K(a) = 5$ $h_K(d) = 4$ $h_K(ad) = 5$
$h_K(b) = 2$ $h_K(ab) = 7$
$h_K(c) = 3$ $h_K(ac) = 4$

Sparsifier G'=(K,E')



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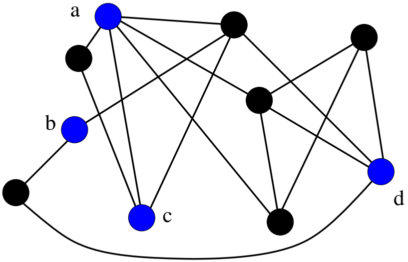
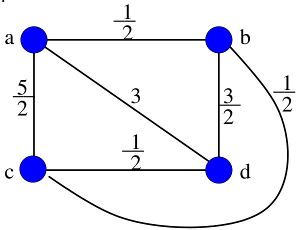
General Approach: Cut Sparsifiers

Quality $= \frac{5}{4}$





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Definition

$G' = (K, E')$ is a **Cut Sparsifier** for $G = (V, E)$ if all cuts in G' are at least as large as the corresponding min-cut in G.

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The **Quality** of a Cut Sparsifier is the maximum ratio of a cut in G' to the corresponding min-cut in G.

Cut Sparsifiers

Good quality Cut Sparsifiers exist!

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For all (undirected) weighted graphs $G = (V, E)$, and all $K \subset V$ there is an (undirected) weighted graph $G' = (K, E')$ such that G' is a $O(\log k/ \log \log k)$-quality Cut Sparsifier.

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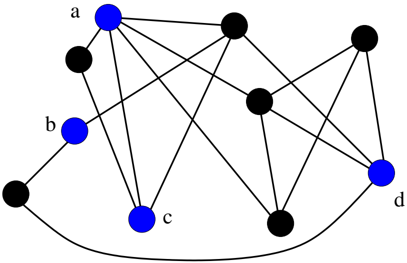
This bound improves to $O(1)$ if G is planar, or if G excludes any fixed minor!

An Application to Steiner Minimum Bisection

Graph $G=(V,E)$



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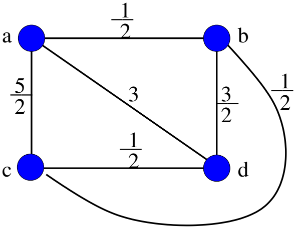

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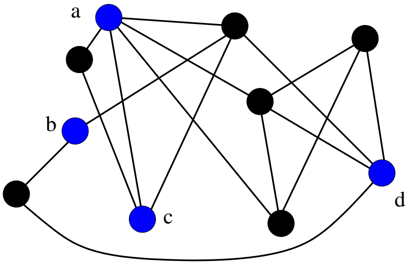

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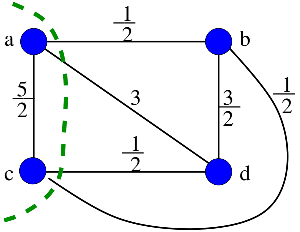

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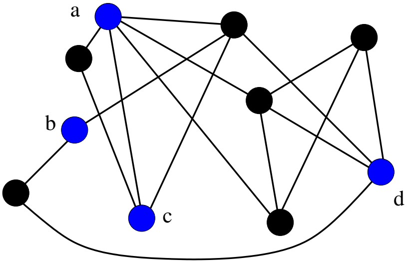

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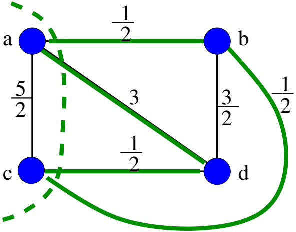

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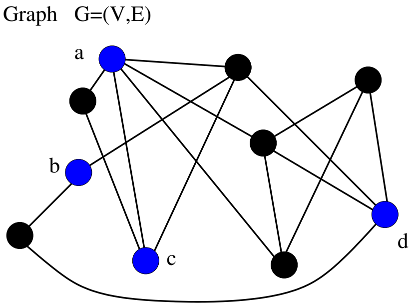

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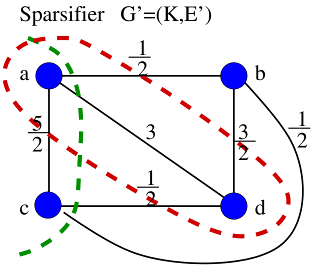

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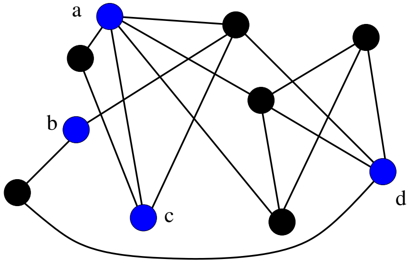

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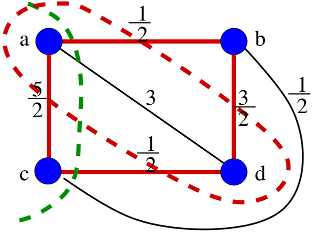

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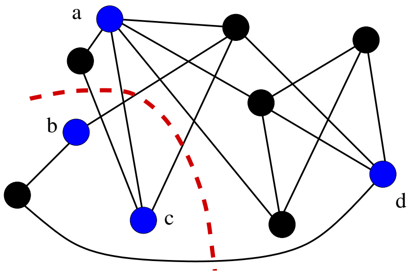

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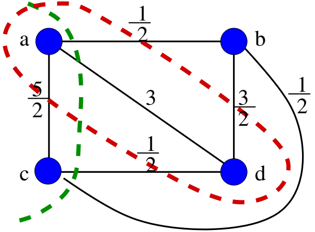

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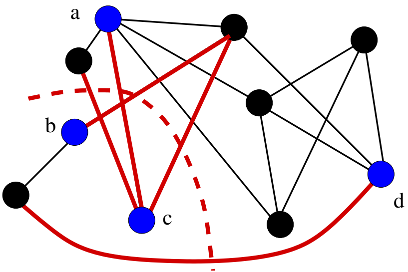

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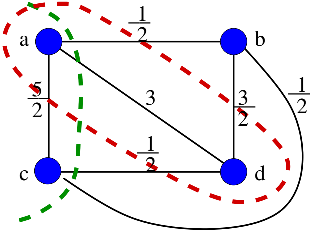

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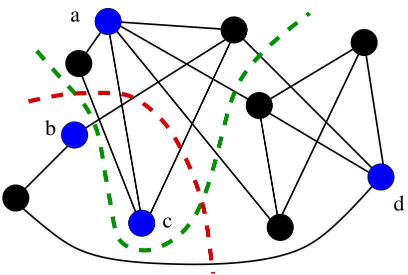

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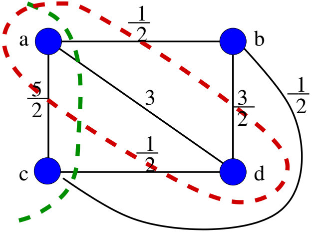

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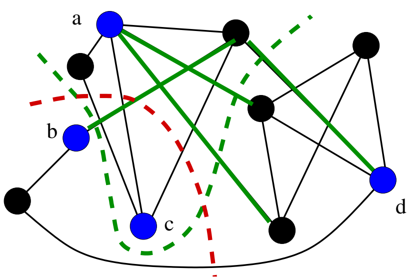

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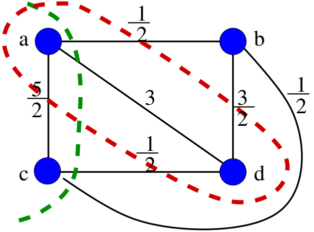

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This will bootstrap a $poly(\log k)$ guarantee from a $poly(\log n)$ guarantee

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Highlights

1. **Approximation Guarantees Independent of the Graph Size:** We give the first *poly*($\log k$) approximation algorithms (or competitive ratios) for: Steiner minimum bisection, requirement cut, l-multicut, oblivious 0-extension, and Steiner generalizations of oblivious routing, min-cut linear arrangement, and minimum linear arrangement

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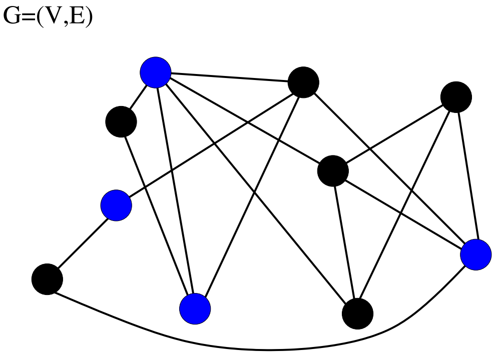
1. **Approximation Guarantees Independent of the Graph Size:** We give the first *poly*$(\log k)$ approximation algorithms (or competitive ratios) for: Steiner minimum bisection, requirement cut, l-multicut, oblivious 0-extension, and Steiner generalizations of oblivious routing, min-cut linear arrangement, and minimum linear arrangement
2. **Oblivious Reductions:** All you need to know about the underlying communication network is its vertex sparsifier

Definition



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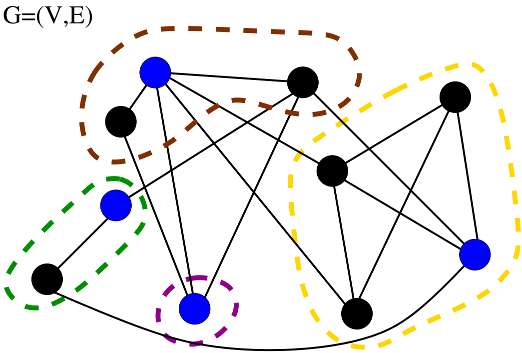

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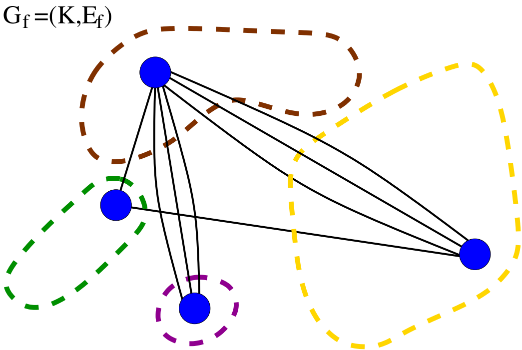
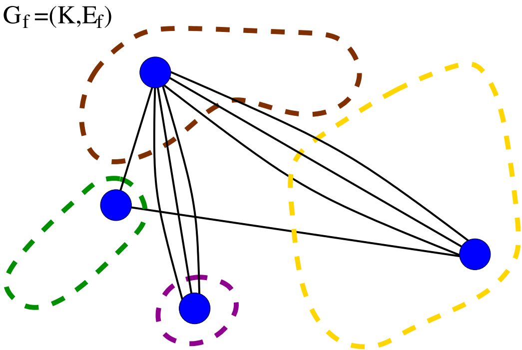


Definition

Let $f : V \rightarrow K$, is a 0-extension if for all $a \in K$, $f(a) = a$.



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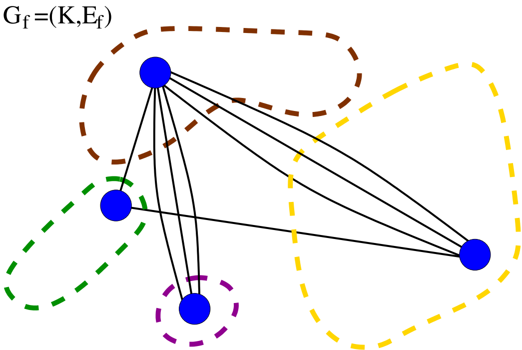

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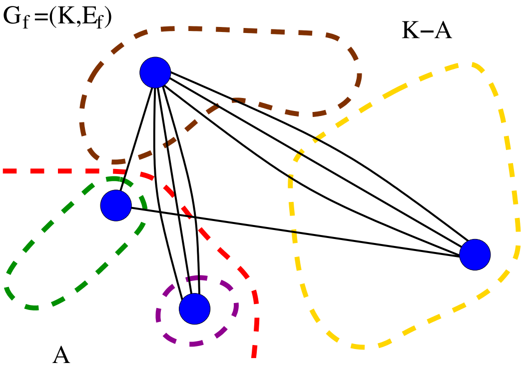

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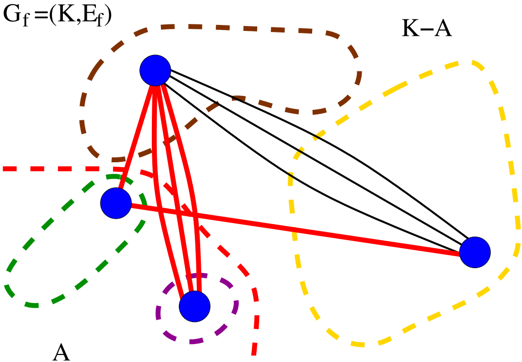

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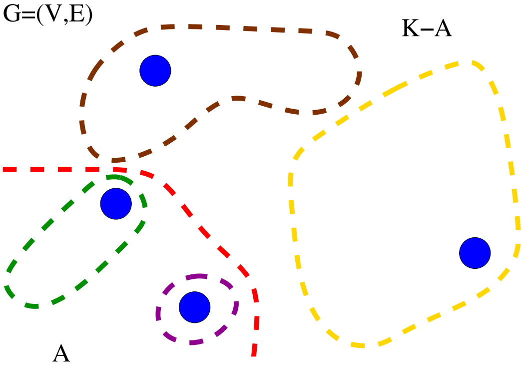

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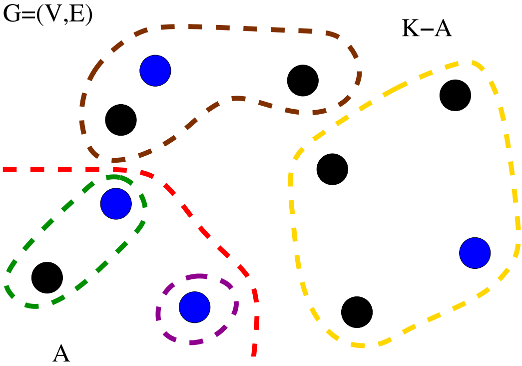

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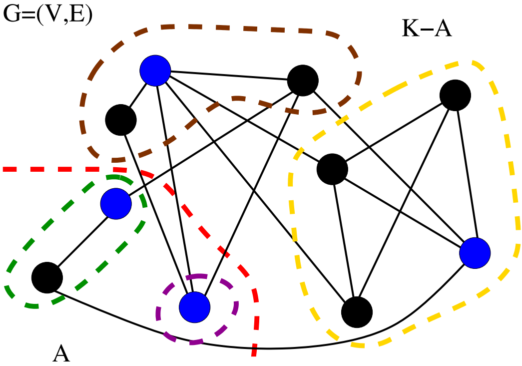

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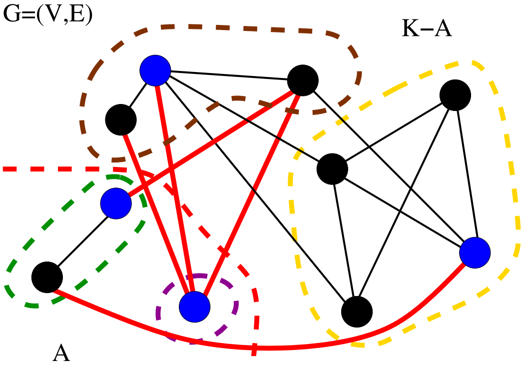
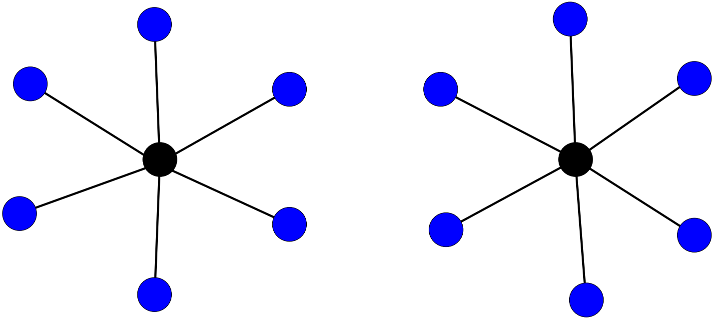


An Example



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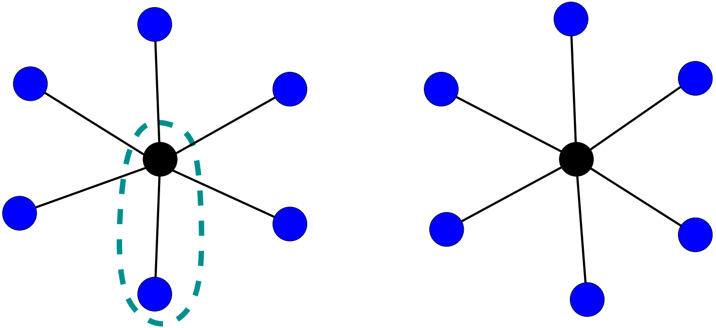

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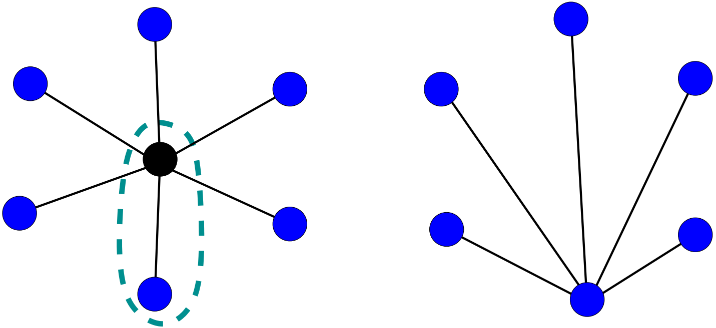

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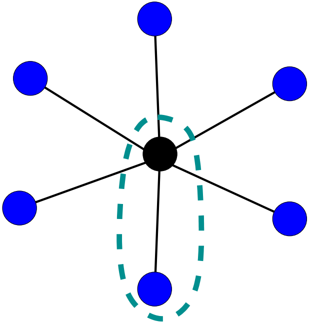
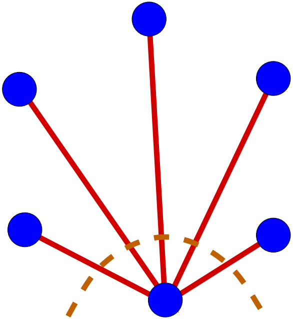


The cost is k–1



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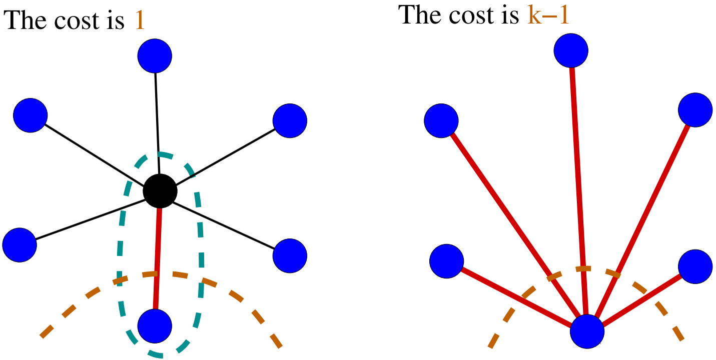

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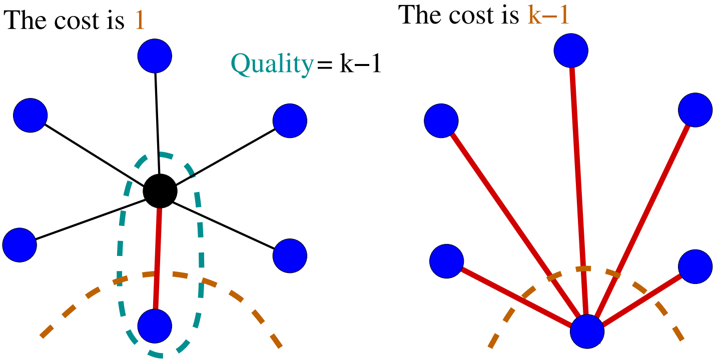

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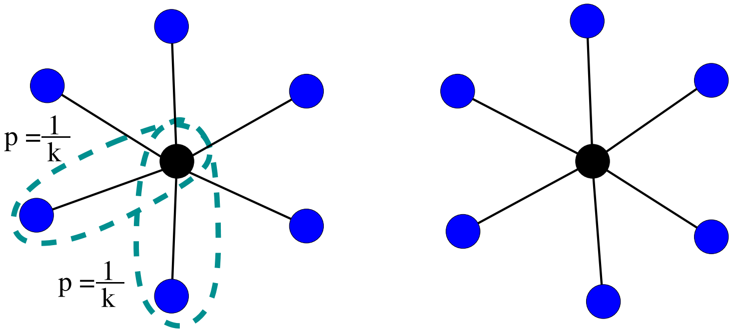

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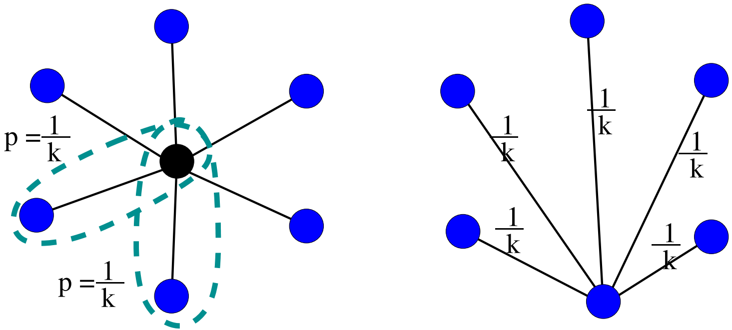

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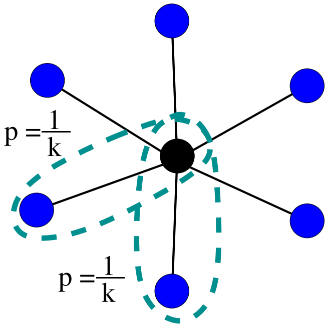

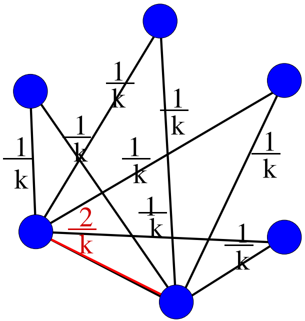

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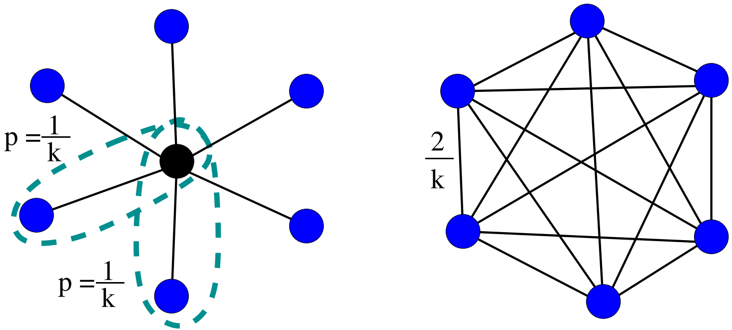

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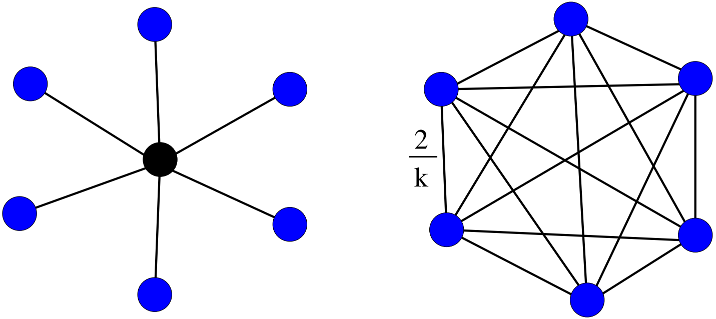

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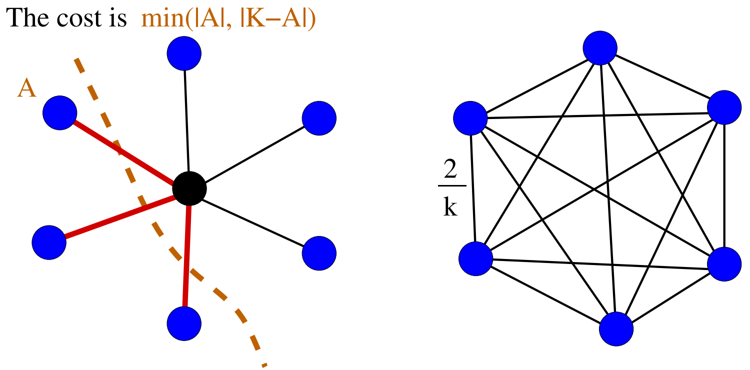
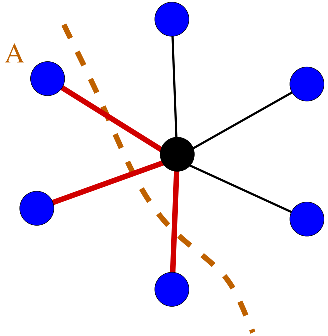
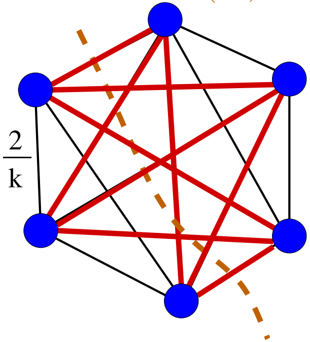


An Example: A Second Attempt

The cost is min(|A|, |K−A|)



The cost is at most 2min(|A|, |K−A|)



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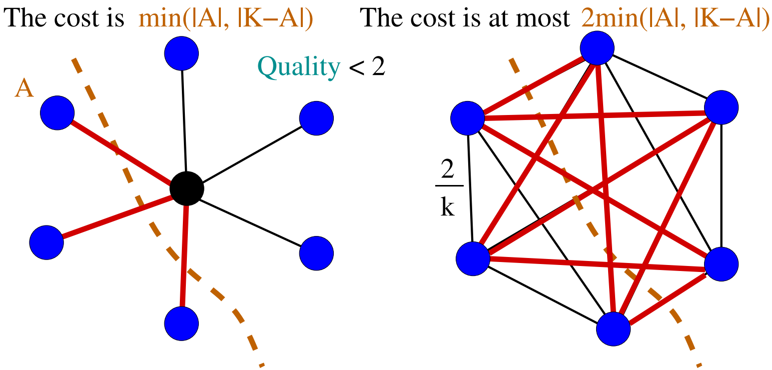

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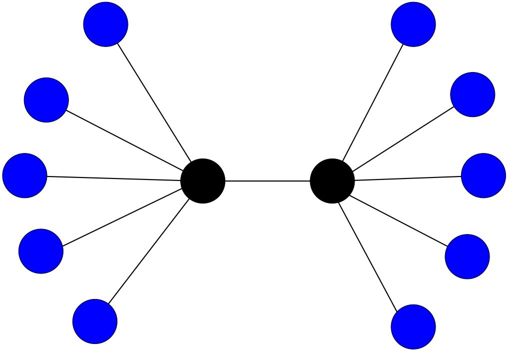

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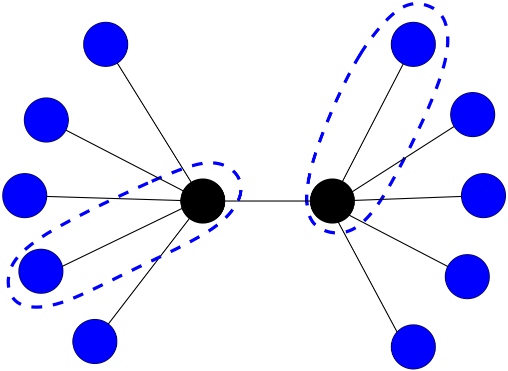

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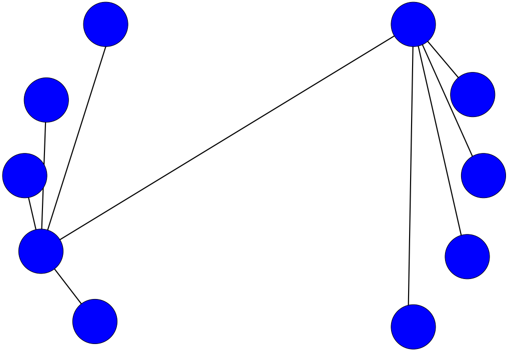

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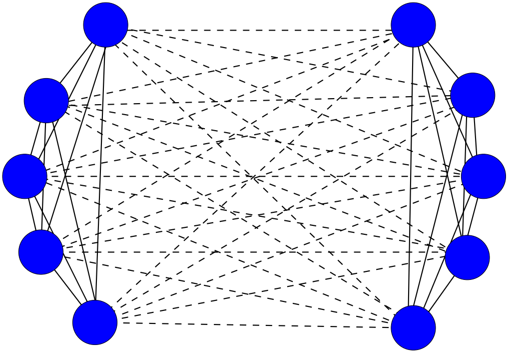
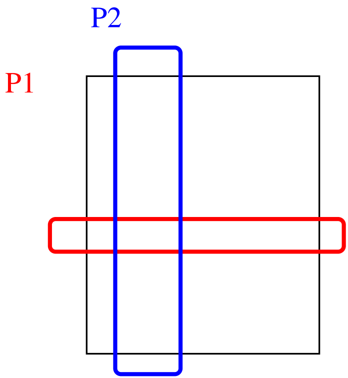


Proof Outline

Proof Outline

1. Define a **Zero-Sum Game**

The Extension-Cut Game



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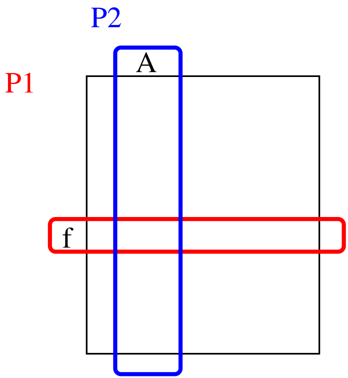

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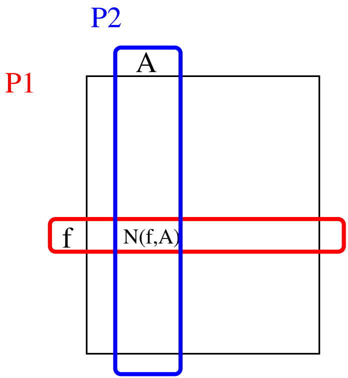

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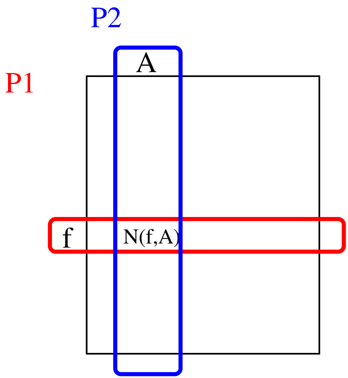

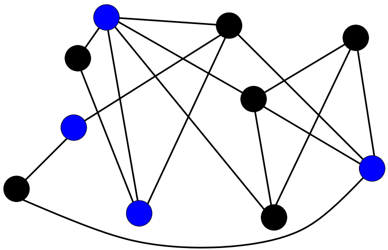

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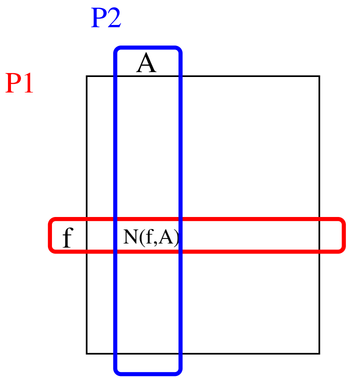

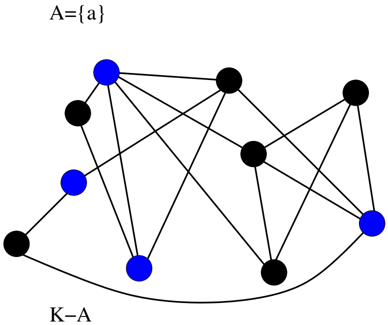

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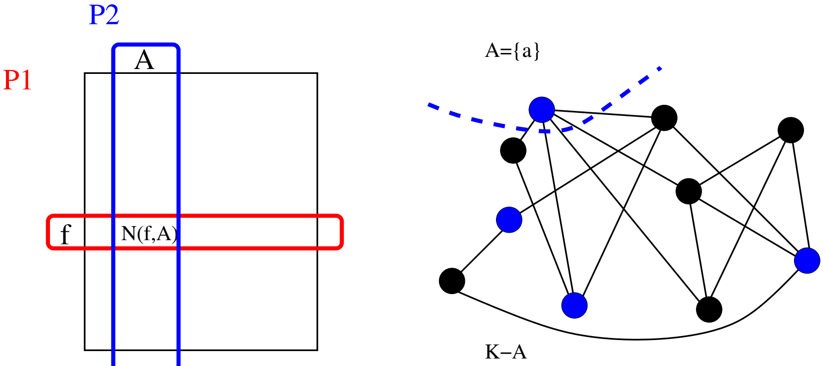

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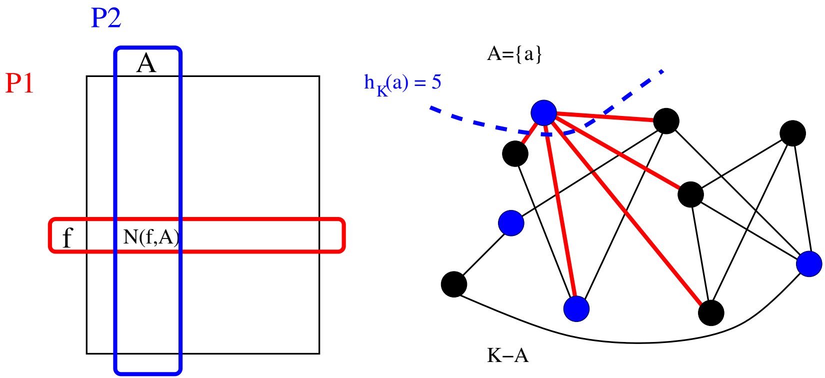

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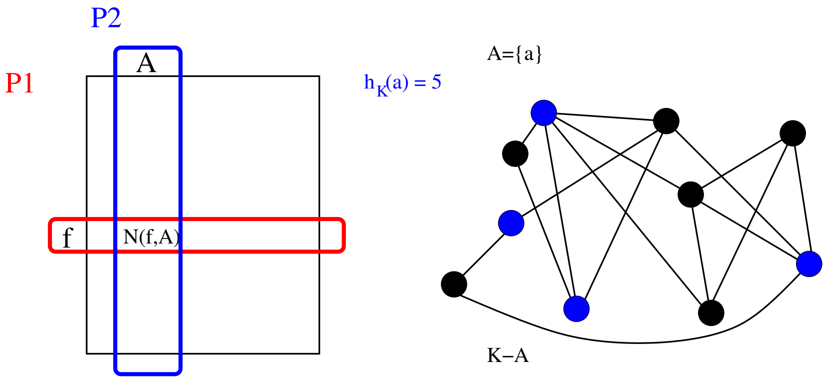

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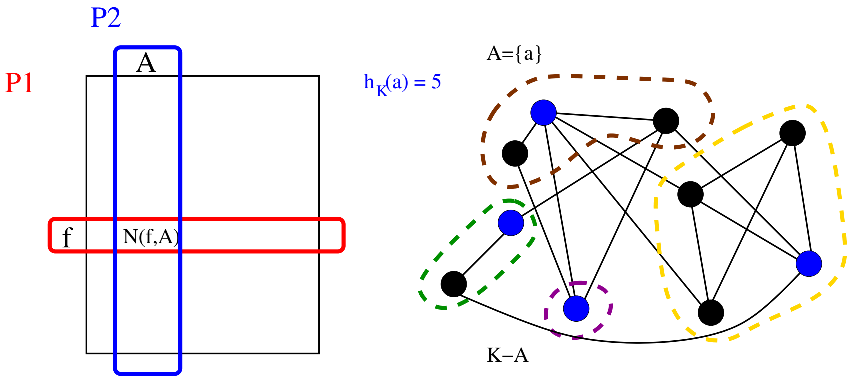

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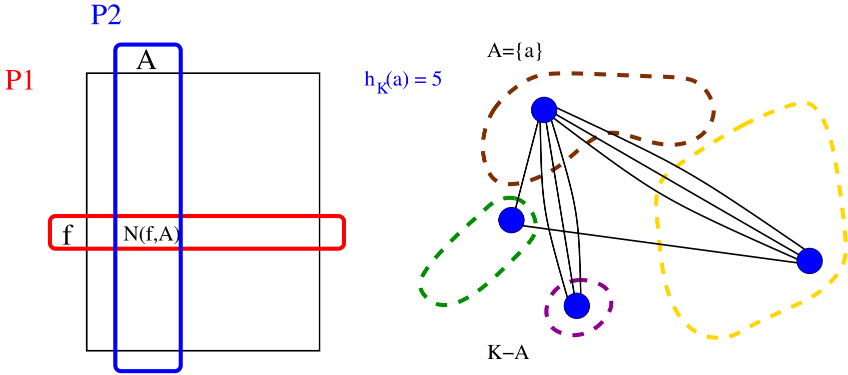

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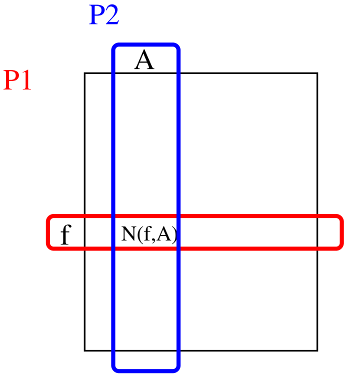

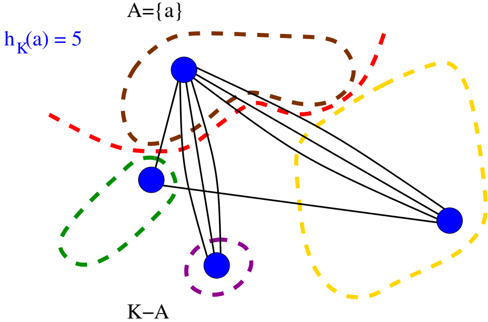

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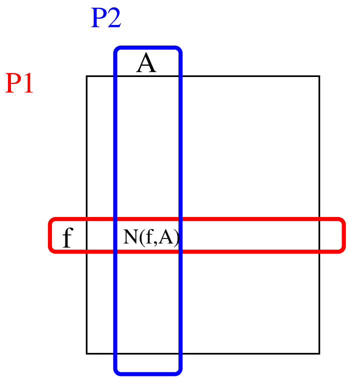

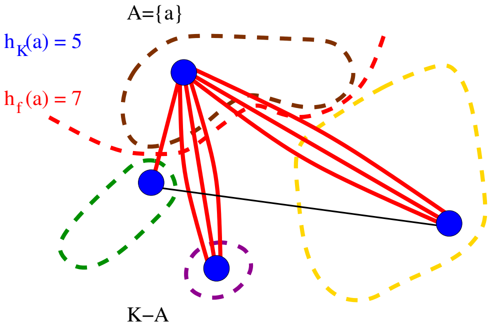

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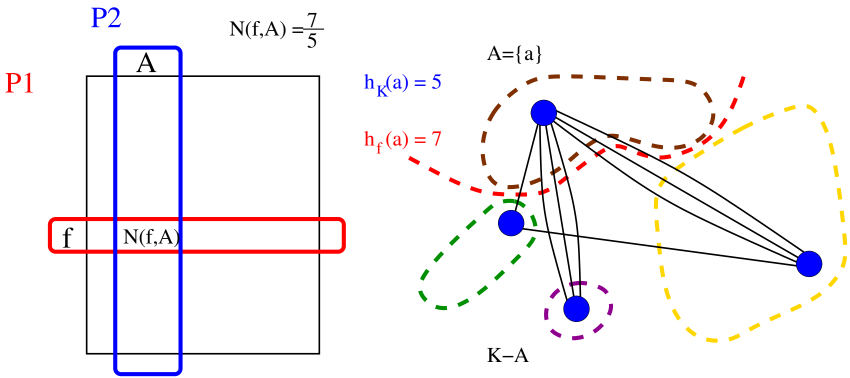

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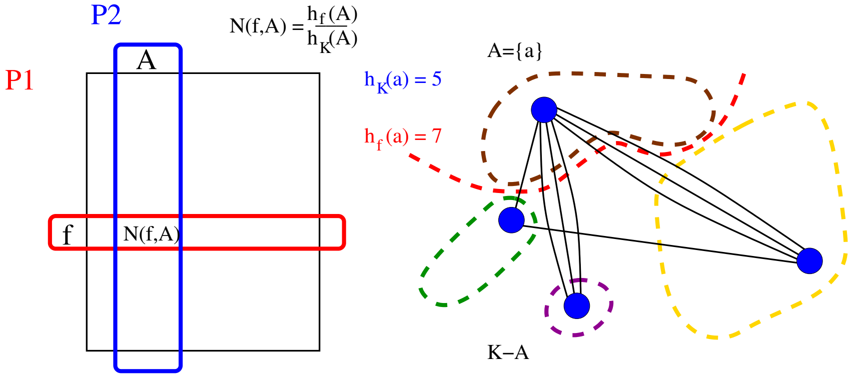

Definition

Let ν denote the game value of the extension-cut game

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So $\exists$ a distribution γ on 0-extensions s.t. for all $A \subset K$:

$$E_{f \leftarrow \gamma}[N(f, A)] \leq \nu$$

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Let $G' = \sum_f \gamma(f) G_f$. Then for all $A \subset K$:

$$h'(A) = \sum_f \gamma(f) h_f(A) = E_{f \leftarrow \gamma}[N(f, A)] h_K(A) \leq \nu h_K(A)$$

Proof Outline

1. Define a **Zero-Sum Game**

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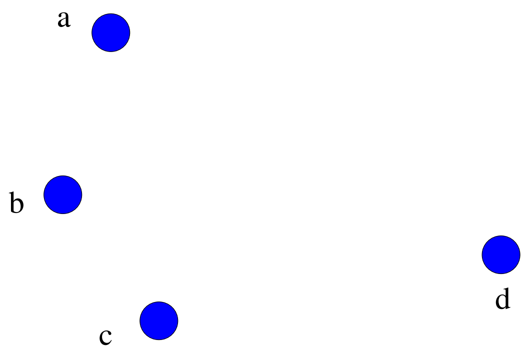
2. The **Best Response** is a 0-Extension Problem

Best Response?

Let $\mu = \frac{1}{2}\{a, b\} + \frac{1}{2}\{a, d\}$



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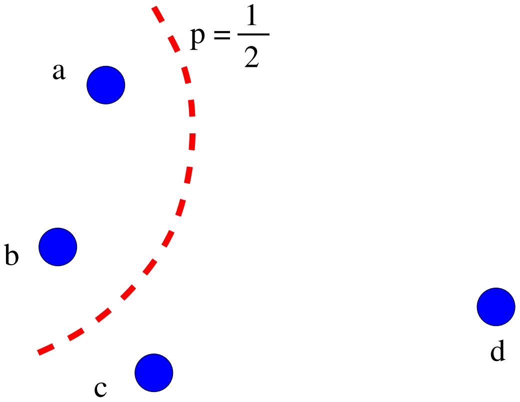

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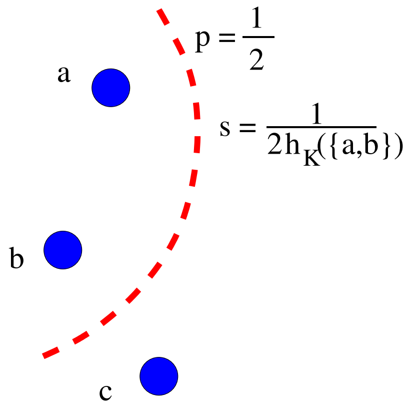

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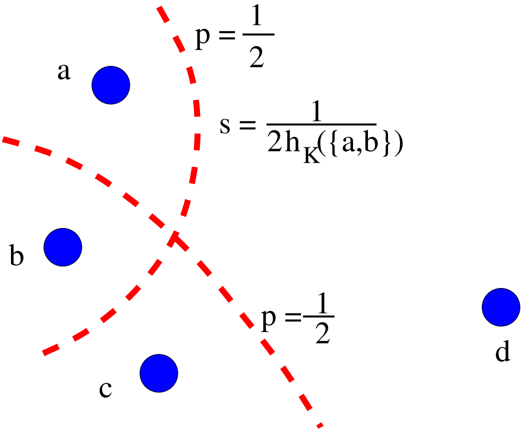

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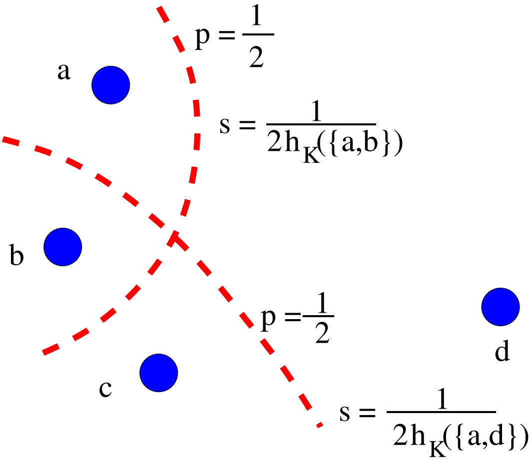

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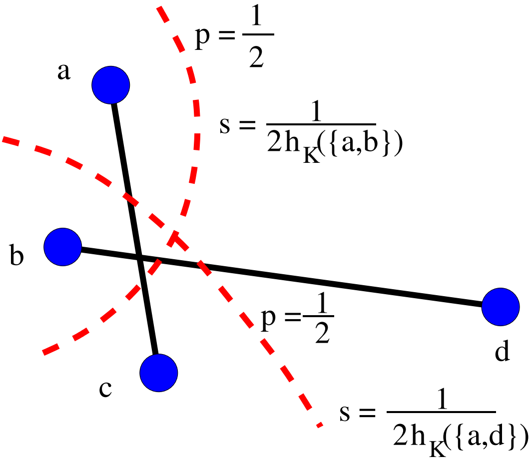

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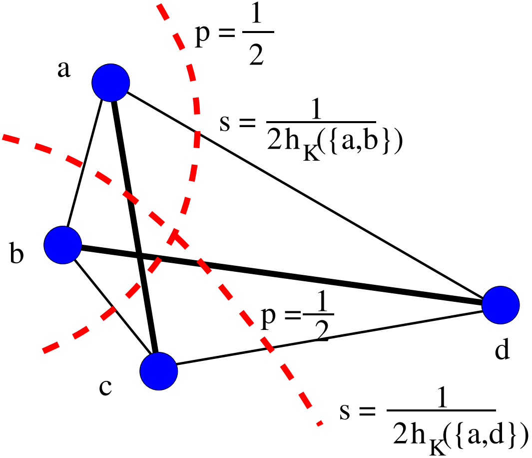

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Bounds on the Integrality Gap

($OPT^* =$ value of the LP)

Theorem (Fakcharoenphol, Harrelson, Rao, Talwar)

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If G excludes any fixed minor,

$$OPT \leq O(1) OPT^*$$

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Epilogue

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1. Constructive results through **lifting**

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4. Separations using **harmonic analysis** of Boolean functions

Thanks!

References

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2. Leighton, Moitra, "Extensions and limits to vertex sparsification", STOC 2010
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