The Power of Tabulation Hashing

Mikkel Thorup

University of Copenhagen

AT&T
Thank you for inviting me to China Theory Week.
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Joint work with Mihai Pătraşcu. Some of it found in Proc. STOC’11.
Target

- Simple and reliable pseudo-random hashing.
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- Providing *algorithmically important* probabilistic guarantees akin to those of truly random hashing, yet easy to implement.
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- Providing algorithmically important probabilistic guarantees akin to those of truly random hashing, yet easy to implement.
- Bridging theory (assuming truly random hashing) with practice (needing something implementable).
Applications of Hashing

Hash tables ($n$ keys and $2n$ hashes: expect 1/2 keys per hash)

- chaining: follow pointers
Applications of Hashing

Hash tables ($n$ keys and $2n$ hashes: expect $1/2$ keys per hash)

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```
   . . . .          a -> t -> x
   . . . .          v
   . . . .        f -> s -> r
```

$x$
Applications of Hashing

Hash tables \((n \text{ keys and } 2n \text{ hashes}: \text{expect } 1/2 \text{ keys per hash})\)

- chaining: follow pointers
- linear probing: sequential search in \(\text{one array}\)

\[
\begin{array}{c}
\bullet \\
q \\
a \\
g \\
c \\
\bullet \\
\bullet \\
t \\
\end{array}
\]
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```
  x ➝
    ➝
    ➝
    ➝
  t
  c ➝
  g ➝
  a ➝
```
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\[
\begin{array}{c}
| a | \\
|   | \\
|   | \\
| y | \\
| w | \\
|   | \\
\end{array}
\quad\rightarrow\quad
\begin{array}{c}
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| s | \\
| z | \\
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Sketching, streaming, and sampling:
- second moment estimation: $F_2(\bar{x}) = \sum_i x_i^2$
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Sketching, streaming, and sampling:

- second moment estimation: $F_2(\bar{x}) = \sum_i x_i^2$
- sketch $A$ and $B$ to later find $|A \cap B|/|A \cup B|$

\[
|A \cap B|/|A \cup B| = \Pr[h(\min h(A)) = \min h(B)]
\]

We need $h$ to be $\varepsilon$-minwise independent:

\[
(\forall) x \notin S : \Pr[h(x) < \min h(S)] = \frac{1 \pm \varepsilon}{|S| + 1}
\]
Applications of Hashing

Hash tables ($n$ keys and $2n$ hashes: expect $1/2$ keys per hash)

- **chaining**: follow pointers.
- **linear probing**: sequential search in *one* array

**Important outside theory.** These simple practical hash tables often bottlenecks in the processing of data—substantial fraction of worlds computational resources spent here.
Carter & Wegman (1977)

We do not have space for truly random hash functions, but

Family $\mathcal{H} = \{h : [u] \rightarrow [b]\}$ $k$-independent iff for random $h \in \mathcal{H}$:

- $(\forall) x \in [u]$, $h(x)$ is uniform in $[b]$;
- $(\forall) x_1, \ldots, x_k \in [u]$, $h(x_1), \ldots, h(x_k)$ are independent.

Prototypical example: degree $k-1$ polynomial

- $u = b$ prime;
- choose $a_0, a_1, \ldots, a_{k-1}$ randomly in $[u]$;
- $h(x) = (a_0 + a_1 x + \cdots + a_{k-1} x^{k-1}) \mod u$. 

Many solutions for $k$-independent hashing proposed, but generally slow for $k > 3$ and too slow for $k > 5$. 
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Many solutions for $k$-independent hashing proposed, but generally slow for $k > 3$ and too slow for $k > 5$. 
How much independence needed?

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Independence has been the ruling measure for quality of hash functions for 30+ years, but is it right?
Simple tabulation

- Simple tabulation goes back to Carter and Wegman’77.
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- Key $x$ divided into $c = O(1)$ characters $x_1, \ldots, x_c$, e.g., 32-bit key as $4 \times 8$-bit characters.
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- For $i = 1, ..., c$, we have truly random hash table:
  $R_i : \text{char} \rightarrow \text{hash values (bit strings)}$
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$$h(x) = R_1[x_1] \oplus \cdots \oplus R_c[x_c]$$
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  $$h(x) = R_1[x_1] \oplus \cdots \oplus R_c[x_c]$$
- Space $cN^{1/c}$ and time $O(c)$. With 8-bit characters, each $R_i$ has 256 entries and fit in L1 cache.
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- Simple tabulation is the fastest 3-independent hashing scheme. Speed like 2 multiplications.
- Not 4-independent: $h(a_1 a_2) \oplus h(a_1 b_2) \oplus h(b_1 a_2) \oplus h(b_1 b_2)$
  $$= (R_1[a_1] \oplus R_2[a_2]) \oplus (R_1[a_1] \oplus R_2[b_2]) \oplus \ldots$$
  $$= (R_1[b_1] \oplus R_2[a_2]) \oplus (R_1[b_1] \oplus R_2[b_2]) = 0.$$
## How much independence needed? Wrong question

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New result: Despite its 4-dependence, simple tabulation suffices for all the above applications:

*One simple and fast hashing scheme for almost all your needs.*
How much independence needed? Wrong question

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Knuth recommends simple tabulation but cites only 3-independence as mathematical quality.
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We prove that dependence of simple tabulation is not harmful in any of the above applications.
Chaining/hashing into bins

**Theorem** Consider hashing $n$ balls into $m \geq n^{1-1/(2c)}$ bins by simple tabulation. Let $q$ be an additional *query ball*, and define $X_q$ as the number of regular balls that hash into a bin chosen as a function of $h(q)$. Let $\mu = \mathbb{E}[X_q] = \frac{n}{m}$. The following probability bounds hold for any constant $\gamma$:

\[
\Pr[X_q \geq (1 + \delta)\mu] \leq \left( \frac{e^\delta}{(1 + \delta)^{(1+\delta)}} \right)^{\Omega(\mu)} + m^{-\gamma}
\]

\[
\Pr[X_q \leq (1 - \delta)\mu] \leq \left( \frac{e^{-\delta}}{(1 - \delta)^{(1-\delta)}} \right)^{\Omega(\mu)} + m^{-\gamma}
\]

With $m \leq n$ bins, every bin gets

\[
\frac{n}{m} \pm O \left( \sqrt{\frac{n}{m} \log^c n} \right)
\]

keys with probability $1 - n^{-\gamma}$. 
Hashing into many bins

**Lemma** If we hash \( n \) keys into \( n^{1+\Omega(1)} \) bins, then all bins get \( O(1) \) keys w.h.p.
Hashing into many bins

**Lemma** If we hash $n$ keys into $n^{1+\Omega(1)}$ bins, then all bins get $O(1)$ keys w.h.p.

Nothing like this lemma holds if we instead of simple tabulation assumed $k$-independent hashing with $k = O(1)$. 
Hashing into many bins

Lemma If we hash \( n \) keys into \( n^{1+\Omega(1)} \) bins, then all bins get \( O(1) \) keys w.h.p.

Proof that for any positive constants \( \varepsilon, \gamma \), if we hash \( n \) keys into \( m \) bins and \( n \leq m^{1-\varepsilon} \), then all bins get less than \( d = 2^{(1+\gamma)/\varepsilon} \) keys with probability \( \geq 1 - m^{-\gamma} \).
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- Reduce $T$ to $T'$ removing all keys $y$ from $T$ with $y_i = a$.
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- The hash of $x$ is independent of the hash of $T'$ as only $h(x)$ depends on $R_i[a]$.
Hashing into many bins

Lemma If we hash $n$ keys into $n^{1+\Omega(1)}$ bins, then all bins get $O(1)$ keys w.h.p.

Proof that for any positive constants $\varepsilon, \gamma$, if we hash $n$ keys into $m$ bins and $n \leq m^{1-\varepsilon}$, then all bins get less than $d = 2^{(1+\gamma)/\varepsilon}$ keys with probability $\geq 1 - m^{-\gamma}$.

Claim 1 Any set $T$ contains a subset $U$ of $\log_2 |T|$ keys that hash independently.

- Let $i$ be the character position where keys in $T$ differ.
- Let $a$ be the least common character in position $i$ and pick $x \in T$ with $x_i = a$
- Reduce $T$ to $T'$ removing all keys $y$ from $T$ with $y_i = a$.
- The hash of $x$ is independent of the hash of $T'$ as only $h(x)$ depends on $R_i[a]$.
- Return $\{x\} \cup U'$ where $U'$ independent subset of $T'$.
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**Claim 2** The probability that there exists $u = (1 + \gamma)/\varepsilon$ keys hashing independently to the same bin is $m^{-\gamma}$. 

▶ There are $(n^u)$ < $n^u$ sets $U$ of $u$ keys to consider.
▶ Each such $U$ hash to one bin with probability $1/m^u-1$.
▶ Probability bound over all $U$ is $n^uu/m^u-1 \leq m^{1-\varepsilon}u + 1 - u = m^{1-\varepsilon}u = m^{-\gamma}$. 


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Basic proof pattern with \( m \geq n^{1-1/(2c)} \) bins
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- Deterministic partition key set \( S \) into groups \( G \) that are mutually “independent”, each of size \( \leq n^{1-1/c} \leq m^{1-\varepsilon} \).
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- If the $X_G$ were really independent, by Chernoff

\[
\Pr[X \geq (1 + \delta)\mu] \leq \left(\frac{e^{\delta}}{(1 + \delta)^{1+\delta}}\right)^{\mu/d}
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Recursive partition into “independent” groups

Define position character \((i, a)\) in key \(x\) iff \(x_i = a\).
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Define position character \((i, a)\) in key \(x\) iff \(x_i = a\).
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Claim \(|G(i, a)| \leq n^{1 - 1/c}\).
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Claim \(|G_{(i,a)}| \leq n^{1-1/c}\).

- For each position \(i \in [c]\), we have \(< n^{1/c}\) characters used by \(> n^{1-1/c}\) keys.
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- So claim false implies \(S\) in hypercube of size \(< (n^{1/c})^c = n\).
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**Claim** \(|G(i,a)| \leq n^{1-1/c}\). □

Recursively, we group \(S \setminus G(i,a)\) and hash all position characters in \(S\) excluding \((i, a)\).
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Now we randomly pick \(R_i[a]\) finalizing hashing of group \(G_{(i,a)}\).
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Now we randomly pick \(R_i[a]\) finalizing hashing of group \(G_{(i,a)}\).
- The contribution \(X_{G_{(i,a)}}\) to our bin is random variable.
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Now we randomly pick $R_i[a]$ finalizing hashing of group $G_{(i,a)}$.
- The contribution $X_{G_{(i,a)}}$ to our bin is random variable.
- The distribution of $X_{G_{(i,a)}}$ depends on previous fixings.
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- The distribution of \(X_{G(i, a)}\) depends on previous fixings.
- But always \(\mathbb{E}[X_{G(i, a)}] = |X_{G(i, a)}|/m\). Moreover \(X_{G(i, a)} \leq d\).
- Good enough for Chernoff bounds.
Chernoff with $m \geq n^{1-1/(2c)}$ bins

W.h.p., the contribution $X$ to given obeys Chernoff

$$\Pr[X \geq (1 + \delta)\mu] \leq \left( \frac{e^\delta}{(1 + \delta)(1+\delta)} \right)^{\mu/d}$$

$$\Pr[X \leq (1 - \delta)\mu] \leq \left( \frac{e^{-\delta}}{(1 - \delta)(1-\delta)} \right)^{\mu/d}$$

Thus, from perspective of chaining, simple tabulation has same type of tail bounds as with truly random hash functions, modulo a constant factor loss and down to polynomially small probabilities.

Similar story for linear probing.
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Similar story for linear probing.
Cuckoo hashing

Each key placed in one of two hash locations.

\[
\begin{array}{c|c}
\text{z} & \bullet \\
\bullet & \bullet \\
y & x \\
x & r \\
\end{array}
\quad
\begin{array}{c|c}
& \bullet \\
& s \\
w & f \\
& a \\
& b \\
\end{array}
\]

\[
x \leadsto x \quad x \leadsto b
\]

**Theorem** With simple tabulation Cuckoo hashing works with probability \(1 - \tilde{\Theta}(n^{-1/3})\).
Cuckoo hashing

Each key placed in one of two hash locations.

```
 x ⇝  x
 z
 ●
 ●
 y
 x
 ●
 r
```

Theorem With simple tabulation Cuckoo hashing works with probability $1 - \tilde{\Theta}(n^{-1/3})$.

- For chaining and linear probing, we did not care about a constant loss, but obstructions to cuckoo hashing may be of just constant size, e.g., 3 keys sharing same two hash locations.

```
●
 s
 w
 f
●
 a
 b
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\[
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\bullet & w \\
\text{y} & f \\
\text{x} & a \\
\bullet & b \\
r & \\
\end{array}
\]

\[
\begin{array}{c|c}
\text{x} \rightsquigarrow & \bullet \\
\bullet & \\
\bullet & \\
\text{x} \rightsquigarrow & \\
\bullet & \\
r & \\
\end{array}
\]

**Theorem** With simple tabulation Cuckoo hashing works with probability \(1 - \tilde{\Theta}(n^{-1/3})\).

- For chaining and linear probing, we did not care about a constant loss, but obstructions to cuckoo hashing may be of just constant size, e.g., 3 keys sharing same two hash locations.
- Very delicate proof showing that obstruction can be used to code random tables \(R_i\) with few bits.
## Speed

<table>
<thead>
<tr>
<th>bits</th>
<th>hashing scheme</th>
<th>32-bit computer</th>
<th>64-bit computer</th>
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</thead>
<tbody>
<tr>
<td>32</td>
<td>univ-mult-shift ((a \times x) &gt;&gt; s)</td>
<td>1.87</td>
<td>2.33</td>
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<tr>
<td>32</td>
<td>2-indep-mult-shift</td>
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<td>2.88</td>
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<tr>
<td>32</td>
<td>5-indep-Mersenne-prime</td>
<td>99.70</td>
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<td>5-indep-TZ-table</td>
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<td>12.66</td>
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<td>simple-table</td>
<td>4.98</td>
<td>4.61</td>
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<tr>
<td>64</td>
<td>simple-table</td>
<td>15.54</td>
<td>11.40</td>
</tr>
</tbody>
</table>

Experiments with help from Yin Zhang.
Robustness in linear probing for dense interval

cumulative fraction vs. average time per insert+delete cycle (nanoseconds)

- simple-table
- univ-mult-shift
- 2-indep-mult-shift
- 5-indep-TZ-table
- 5-indep-Mersenne-prime
Pitch for theory in case of linear probing

- Multiplicative hashing used in practice, but turns out to be very unreliable under typical denial-of-service (DoS) attacks based on consecutive IP addresses: systematic good performance 95% of the time, but systematic terrible performance 5% of the time [TZ’10].

- Problems in randomized algorithms like hashing hard to detect for practitioners. Hard for them to know if bad performance is from being unlucky, or because of systematic problems.

- Linear probing had gotten a reputation for being fastest in practice, but sometimes unreliable needing special protection against bad cases.

- Here we proved linear probing safe with good probabilistic performance for all input if we use simple tabulation.

- Simple tabulation also powerful for chaining, cuckoo hashing, and min-wise hashing: one simple and fast scheme for (almost) all your needs.
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Work in progress: twisted tabulation

- With chaining and linear probing, each operation takes expected constant time, but out of $\sqrt{n}$ operations, some are expected to take $\tilde{\Omega}(\log n)$ time.

- With truly random hash function, we handle every window of $\log n$ operations in $O(\log n)$ time w.h.p.

- Hence, with small buffer (as in Internet routers), we do get down to constant time per operation!

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Implementing Chernoff bounds with twisted tabulation

- 0-1 variables $X_i$ where $X_i = 1$ with probability $p_i$. 

Exponential concentration of $X = \sum_i X_i$ around mean.

Application: trust polynomial number of logarithmic estimates with high probability—the log factor in many randomized algorithms.

With hashing into $[0, 1]$, set $X_i = 1$ if $h(i) < p_i$.

With bounded dependence only polynomial concentration.

With twisted tabulation: for any constant $\gamma$,

$$\Pr[X \geq (1 + \delta) \mu] \leq \left( e^{\delta} (1 + \delta) (1 + \delta) \right)^{-\Omega(\mu)} + \sum_{-\gamma} \Pr[X \leq (1 - \delta) \mu] \leq \left( e^{-\delta} (1 - \delta) (1 - \delta) \right)^{-\Omega(\mu)} + \sum_{-\gamma}$$

With simple tabulation, additive term $(\max_i p_i) \gamma$—in the hash tables we had $p \approx 1/n$. 

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- So far, no technique is known that can make any such separation between deterministic and randomized solutions for any data structure problem.