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Market Equilibrium under Separable, Piecewise-Linear, Concave Utilities

Vijay V. Vazirani[1] Mihalis Yannakakis[2]
[1]Georgia Institute of Technology, Atlanta, GA 30332–0280.
[1]Computer Science Department, Columbia University, New York, NY 10027.
vazirani@cc.gatech.edu mihalis@cs.columbia.edu

Abstract: We consider Fisher and Arrow-Debreu markets under additively-separable, piecewise-linear, concave utility functions, and obtain the following results:

- For both market models, if an equilibrium exists, there is one that is rational and can be written using polynomially many bits.
- There is no efficiently checkable necessary and sufficient condition for the existence of an equilibrium: The problem of checking for existence of an equilibrium is NP-complete for both market models; the same holds for existence of an ϵ-approximate equilibrium, for $\epsilon = O(n^{-5})$.
- Under standard (mild) sufficient conditions, the problem of finding an exact equilibrium is in PPAD for both market models. We note that this is the first result showing membership in PPAD for a market model defined by an important, broad class of utility functions.
- Finally, building on the techniques of [3] we prove that under these sufficient conditions, finding an equilibrium for Fisher markets is PPAD-hard.

Keywords: market equilibrium; piecewise-linear utility functions; PPAD

1 Introduction

The following was the central question within mathematical economics for almost a century: Does a complex economy, with numerous goods and a large number of agents with diverse desires and buying powers, admit equilibrium prices? Its study culminated in the celebrated Arrow-Debreu Theorem [1] which provided an affirmative answer under some assumptions on the utility functions (they must satisfy non-satiation and be continuous and quasi-concave) and initial endowments of the agents (each agent must have a positive amount of each commodity); these are called *standard sufficient conditions*. Over the years, milder sufficient conditions were obtained for the existence of equilibrium, see e.g. [20] and the references therein. In some restricted cases, the sufficient conditions were also found to be necessary, i.e., they characterized the existence of equilibria in the corresponding markets.

Besides existence, another fundamental question is efficient computability of equilibria. We note that the proof of the Arrow-Debreu Theorem was based on the Kakutani's fixed point theorem and alternative proofs are based on Brouwer's theorem; they are all therefore highly non-constructive. In fact, theorems proving the existence of market equilibria and the existence of fixed points are closely related and in a sense equivalent: for excess demand functions that satisfy standard conditions, the existence of an equilibrium can be derived from Brouwer's theorem, and conversely Brouwer's theorem, for general continuous functions, can be derived from the equilibrium theorem [25]; the sufficient conditions on an excess demand function are continuity, homogeneity and Walras' Law, i.e., that the inner product of the price vector and the excess demand function vector be zero. Furthermore, by the Sonnenschein-Mantel-Debreu theorem, all functions satisfying these standard conditions for excess demand functions can be realized by suitable utility functions.

Scarf [23] initiated the development of algorithms for computing market equilibria, introducing a family of procedures that compute approximate price equilibria by pivoting in a simplicial subdivision of the price simplex. A number of other methods, including Newton-based, homotopy methods, etc., have been developed in the following decades. These algorithms perform well in practice for several markets, but their running time is not polynomially bounded. The study of efficient computability of equilibria, from the perspective of modern theory of computation, was initiated by Megiddo [21] and Papadimitriou [22].

In recent years there has been a surge of interest in understanding computability of market equilibria, which is in part motivated by possible applications to markets on the Internet. This study has concentrated on the two fundamental market models of Fisher [2] and Arrow-Debreu [1] (the latter is also known as the Walrasian model or the exchange model, and is more general than the Fisher model) under increasingly general and realistic utility functions. For each class of utility functions, two main algorithmic questions arise: (1) Can we determine necessary and sufficient conditions for the existence of an equilibrium? A good characterization should be efficiently checkable, hence the question can be phrased algorithmically as: What is the complexity of checking for existence of an equilibrium? (2) If suitable sufficient conditions have been established for the existence of an equilibrium, what is the complexity of finding an equilibrium for an instance satisfying these conditions?

In a general setting, e.g., for markets satisfying standard sufficient conditions, and specified by demand functions given by polynomial-time Turing machines or by explicit algebraic formulae, the computation of equilibria is (apparently) hard [13, 22]. To have any hope of efficient algorithms, we need to restrict the class of demand/utility functions. Several important classes of functions have been studied over the years.

Not surprisingly, the first results were for linear utility functions [14]. If the input parameters are rational (as is standard in computer science), then there is always a rational equilibrium for this case and there are simple, efficiently checkable necessary and sufficient conditions for the existence of an equilibrium; for the Fisher model, the conditions are straightforward, and for the Arrow-Debreu model, they were given by Gale [15]. Moreover, for instances satisfying these conditions, polynomial time algorithms were obtained for finding equilibria [11, 18].

Complexity results were also obtained for some specific non-linear utility functions that are well-studied in economics, e.g., Cobb-Douglas, CES, and Leontief; the last case is particularly relevant to this discussion. For this case, the equilibria are in general irrational for both market models [5, 12]. For the Fisher model, assuming suitable sufficient conditions, the problem of approximately computing an equilibrium is polynomial time solvable [5, 26]. For the Arrow-Debreu model, checking existence of an equilibrium is NP-hard, and for instances satisfying the standard Arrow-Debreu sufficient conditions, the computation of approximate equilibria is PPAD-hard [6, 9, 17]. Note that these are hardness, rather than completeness, results because these problems for Leontief markets not lie necessarily in NP and PPAD. Also note the difference in the complexities of the two market models.

Within economics, concave utilities occupy a special place, since they capture the natural condition of decreasing marginal utilities. Hence, resolving their complexity has taken center stage over the last few years. Since we are dealing with a discrete computational model, it is natural to consider piecewise-linear, concave utilities. These can be further divided into two cases, non-separable and additively separable over goods; clearly, the latter is a subcase of the former. The non-separable case contains Leontief utilities and so the hardness results mentioned above for the Arrow-Debreu model carry over to this case. However, if the number of goods is a constant, then a polynomial time algorithm exists for both market models [10].

This leaves the case of additively separable piecewise-linear, concave utility functions. Recently, Chen, Dai, Du and Teng [3] made a breakthrough on this question by showing PPAD-hardness of computing equilibria, even approximate equilibria, for Arrow-Debreu markets with such utilities[1].

Our results for this class of utility functions are summarized below.

- For both market models, if an equilibrium exists, there is one that is rational and can be written using polynomially many bits.
- There is no efficiently checkable necessary and sufficient condition for the existence of an equilibrium: The problem of checking for existence of an equilibrium is NP-complete for both market models; the same holds for existence of an ϵ-approximate equilibrium, for $\epsilon = O(n^{-5})$.
- Under standard (mild) sufficient conditions, the problem of finding an exact equilibrium is in PPAD for both market models. We note that this is the first result showing membership in PPAD for a market model defined by an important, broad class of utility functions.
- Finally, building on the techniques of [3] we prove that under these sufficient conditions, finding an equilibrium for Fisher markets is PPAD-hard.

Observe that, unlike the Leontief case, the two market models turn out to have the same complexity in this case.

We remark that two of these results were obtained independently and concurrently by other authors: rationality was also proven by Devanur and Kannan for both market models [10] and PPAD-hardness for Fisher markets was proven by Chen and Teng [7] (as noted in both these papers).

We also remark that in a recent paper, Ye

[1]Their initial claim, that the problem of finding an approximate equilibrium lies in PPAD, has been recently rescinded.

showed a distinction between Fisher and Arrow-Debreu markets for a related, though different, model of piecewise-linear concave utility functions; in particular, he showed that the Fisher case can be solved in polynomial time, whereas the Arrow-Debreu case is equivalent to solving a linear complementarity problem [26]. In Section 2 we will explain how his model is different from ours.

How does the "invisible hand of the market," in Adam Smith's famous words, find equilibria? The intractability results of [3, 7] and the current paper make this question even more mysterious.

1.1 Techniques Used

Our results involve several novel techniques; below we give an overview primarily for the positive results (the first and third results in the list in the Abstract).

The combinatorial algorithm for Fisher's linear case [11] gave new insights into the combinatorial structure underlying equilibrium prices and allocations. Given prices $\boldsymbol{p}$, [11] showed how to construct a suitable network such that a max-flow in it helped determine if $\boldsymbol{p}$ are equilibrium prices.

We first extend this structure to the case of separable, piecewise-linear, concave utilities; the main difference being that in this case, in general, at given prices $\boldsymbol{p}$, a buyer's optimal bundle *must* include certain quantities of certain goods – these are called *forced allocations*. The money that is left over after buying forced allocations is to be spent on buying *flexible allocations* from a suitable subset of goods with specified upper bounds on quantity, and *any* allocation exhausting the left-over money leads to an optimal bundle.

Our network is also a function of prices $\boldsymbol{p}$ and incorporates information about forced allocations and the choices available for flexible allocations. Again, a max-flow in this network helps determine if $\boldsymbol{p}$ are equilibrium prices (see Lemma 1). The problem of finding a max-flow in this network can be written as an LP in a straightforward manner.

The next transformation is the most interesting. We assume that prices $\boldsymbol{p}$ are now variables and the network is constructed for a *guess* on forced allocations and choices available for flexible allocations. It turns out that all edge capacities in this network are linear functions of the price variables. Moreover, max-flow in this network, which is a function of prices, can still be written as an LP. We then show if the guess is good, i.e., corresponds to an equilibrium, then the optimal solution to this new LP gives the corresponding equilibrium prices and allocations. Since the solution to an LP is rational, the theorem follows.

Because of rationality, equilibria for these markets can be computed exactly and this leads to the possibility that these problems may lie in PPAD, under suitable sufficient conditions. We show that this is indeed the case for both market models; this is the technically most involved result of our paper.

There are very few ways for showing membership in PPAD. A promising approach for our case is to use the characterization of PPAD of [13] as the class of exact fixed point computation problems for piecewise-linear, polynomial time computable Brouwer functions. The Brouwer functions that have been proposed for market equilibria, such as those of Geneakoplos and McKenzie, are the obvious candidates. Unfortunately, we do not see how to do this: Although it is possible to show that these functions are polynomial time computable (this is nontrivial, e.g., for the Geneakoplos function), it is not clear how to transfer the piecewise-linearity of the utility functions to the Brouwer function.

Another approach is to reduce the problem to the computation of an approximate fixed point for a suitable general (not necessarily piecewise-linear) Brouwer function F that satisfies three conditions: (i) it is polynomially continuous (for example, Lipschitz continuous for a Lipschitz constant that is $O(2^{poly(n)})$), (ii) it is polynomial-time computable, and (iii) any weakly approximate fixed point of the function can be used to efficiently obtain a desired solution, e.g., a price equilibrium in our case (see [13] for a proof). By a *weakly approximate fixed point* we mean a point x such that $|F(x)-x|$ is small. However, such a point may be far from all the fixed points, and this makes task (iii) challenging[2].

The task is further complicated by the fact that, for given prices, the demand, i.e., optimal bundle, of an agent is in general not unique, i.e., it is a correspondence and a not function. Furthermore, this correspondence is very sensitive to the prices – an extremely small change in prices may lead to drastic changes in the demand.

Instead, we employ a combination of the two approaches. Let $\mathcal{M}$ be an instance of a market in the class defined above. We start with the correspondence F of a Kakutani Theorem-based proof of existence of equilibrium for $\mathcal{M}$; this is a correspondence on pairs of price and allocation vectors, (p, x), such that the price components of its fixed points correspond to the set

[2]For example, this is the reason that we cannot place in PPAD the approximation of Nash equilibria in 3-player games. If we could do this, then this would have other important consequences; e.g., it would resolve the longstanding open problem of determining whether the square root sum problem is in NP [13].

of price equilibria for $\mathcal{M}$. We next obtain a piecewise-linear Brouwer function G that approximates F. The function G is easily computable, and hence finding an exact fixed point, (p^*, x^*), for it is in PPAD, by the characterization of PPAD given in [13]. Clearly, (p^*, x^*) may not be a fixed point of F. In addition, it may not even be close to any fixed point of F.

The heart of the proof lies in showing how to efficiently compute a price equilibrium p' for $\mathcal{M}$ from the fixed point (p^*, x^*) of G. For this, we show several properties of the fixed point (p^*, x^*) that allow us to identify which allocations should be forced and which flexible in an equilibrium, i.e., to pin down the combinatorial essence of the problem. We set up an LP, similar to the one used for proving rationality for the specification of flexible and forced allocations derived from (p^*, x^*), but with the constraints relaxed by a variable error amount ϵ. The objective function of the LP is to minimize ϵ. We use the properties of the fixed point (p^*, x^*) to show that it induces a feasible solution to the LP with a very small value of $\epsilon = 2^{-2m}$, where m is a parameter of the market instance $\mathcal{M}$ that upper bounds the bit complexity of an optimal solution to the LP, i.e., the size of the LP and bounds on its coefficients imply that the optimal solution to it must be either 0 or at least 2^{-m}; hence it must be zero. Therefore, solving the LP gives us an exact price equilibrium for market $\mathcal{M}$, say p'. Note that the entire computation involves finding a fixed point of G, a piecewise-linear Brower function, followed by a polynomial time computation. Since this can all be accomplished in PPAD, we get the desired membership result.

Observe that the function G is a Brouwer function, so it has a fixed point (p^*, x^*) regardless of whether the given market has an equilibrium or not. Obviously we cannot derive from (p^*, x^*) a market equilibrium if there is none, so the proof of correctness for the constructed price vector p' has to crucially use the fact that the given market instance satisfies the standard sufficient conditions for the existence of an equilibrium. Moreover, the proof must simultaneously show (constructively, in polynomial time) their sufficiency. Can we expect this procedure and proof to work for all piecewise-linear markets that have an equilibrium, i.e., even ones not satisfying the sufficient conditions? In view of the NP-completeness of the existence problem, the answer is “No”; indeed, if this were the case, then NP would be contained in PPAD, which would imply NP=coNP.

We comment briefly on the negative results (the second and fourth results). We exploit the fact that the high sensitivity of the demands (optimal bundles) to small changes in prices can be combined with well-chosen “pieces” of the piecewise-linear utility functions to give the problems a discrete feel: an agent either buys a segment of a good completely or not at all, depending on how the prices of goods compare with each other. With a careful encoding, this discreteness can be reflected in the choices of the prices in the potential equilibria.

2 Fisher’s Model with Piecewise-linear, Concave Utilities

Fisher’s market model [2] is the following. Let G be a set of divisible goods and B be a set of buyers, $|G| = g$, $|B| = n$. Assume that the goods are numbered from 1 to g and the buyers are numbered from 1 to n. Each buyer $i \in B$ comes to the market with a specified amount of money, say $e(i) \in \mathbf{Q}^+$ dollars. We will assume w.l.o.g. that the amount of each good available is unit. For each buyer i and good j we are specified a function $f_j^i : \mathbf{R}^+ \to \mathbf{R}^+$ which gives the utility that i derives as a function of the amount of good j that she receives. Her overall utility, $u_i(x)$ for a bundle $x = (x_1, \ldots, x_g)$ of goods is additively separable over the goods, i.e., $u_i(x) = \sum_{j \in G} f_j^i(x_j)$. Let $M = \sum_{i \in B} e(i)$ denote the total money of all buyers.

In this paper, we will deal with the case that the f_j^i’s are (non-negative) non-decreasing piecewise-linear, concave functions. Given prices $\boldsymbol{p} = (p_1, \ldots, p_g)$ for all the goods, consider bundles (baskets) of goods that make each buyer i happiest (there could be many such bundles). We will say that $\boldsymbol{p}$ are *equilibrium prices* if there are choices of optimal bundles for the buyers, such that after each buyer is given an optimal bundle, all the money is spent and there is no deficiency or surplus of any good, i.e., the market clears.

Remark: Ye uses a somewhat different model of piecewise-linear concave utility functions in [26]. Specifically, the utility function $u_i(x)$ of buyer i for a bundle $x = (x_1, \ldots, x_g)$ of goods is a function of the form $\min_k u_i^k(x)$, where each $u_i^k(x)$ is a homogeneous, linear function of the form $u_i^k(x) = \sum_{j \in G} u_{ij}^k x_j$. A utility function $u_i(x)$ in our model can be expressed as the minimum of a set of linear functions, but (i) an exponential number of functions will be needed in general, and (ii) the functions are not homogeneous.

We will call each piece of f_j^i a *segment*. The set of segments defined in function f_j^i will be denoted $\text{seg}(f_j^i)$. The slope of a segment specifies the rate at which the buyer derives utility per unit of good received. Suppose one of these segments, s, has range $[a, b] \subseteq \mathbf{R}^+$, and a slope of c. Then, we will define amount$(s) = b - a$, slope$(s) = c$, and good$(s) = j$. We will assume that for each segment s specified in the problem instance, slope(s) and amount(s) are rational numbers. Let segments(i) denote the set of all segments of buyer i, i.e., segments$(i) = \bigcup_{j=1}^{g} \text{seg}(f_j^i)$.

The following (mild) condition suffices for the existence of an equilibrium, as shown in Section 4.

$$\forall i \in B \ \exists j \in G : \sum_{s \in \text{seg}(f_j^i), \text{slope}(s) > 0} \text{amount}(s) > 1$$

3 Rationality of Equilibrium Prices

Given an instance $\mathcal{M}$ of Fisher's market with piecewise-linear, concave utilities and prices $\boldsymbol{p}$ of goods, we first show how to determine if $\boldsymbol{p}$ constitute equilibrium prices. We will assume that $\boldsymbol{p}$ satisfies the condition that the sum of prices of all goods equals the total money of the buyers, i.e., $\sum_j p_j = \sum_i e(i)$.

3.1 Bang per Buck, Allocations, and the Network

Given nonzero prices $\boldsymbol{p} = (p_1, \ldots, p_g)$, we characterize optimal baskets for each buyer relative to $\boldsymbol{p}$. Define the *bang per buck* relative to prices $\boldsymbol{p}$ for segment $s \in \text{seg}(f_j^i), j \neq 0$, to be bpb$(s) = \text{slope}(s)/p_j$. Sort all segments $s \in$ segments(i) by decreasing bang per buck, and partition by equality into classes: $Q_1, Q_2, \ldots$. If for segment s, good$(s) = j$, then the *value of segment* s, value$(s) = \text{amount}(s) \cdot p_j$. For a class Q_l, define value(Q_l) to be the sum of the values of segments in it. At prices $\boldsymbol{p}$, goods corresponding to segments in Q_l make i equally happy, and those in Q_l make i strictly happier than those in Q_{l+1}.

Find $k_i \geq 1$ such that $\sum_{l<k_i} \text{value}(Q_l) \leq e(i) < \sum_{1 \leq l \leq k_i} \text{value}(Q_l)$. At prices $\boldsymbol{p}$, i's optimal allocation must contain goods corresponding to all segments in $Q_1, \ldots, Q_{k_i-1}$, and a bundle of goods worth $e(i) - (\sum_{1 \leq l \leq k_i - 1} \text{value}(Q_l))$ corresponding to segments in Q_{k_i}. We will say that for buyer i, at prices $\boldsymbol{p}$, $Q_1, \ldots, Q_{k_i-1}$ are her *forced partitions*, Q_{k_i} is her *flexible partition*, and $Q_{k_i+1}, \ldots$ are her *undesirable partitions*. Similarly, segments in these three sets will be called *forced, flexible and undesirable segments*, respectively.

For buyer i, we will denote the amount of money spent on forced segments by spent$(i) = \sum_{l<k_i} \text{value}(Q_l)$. Define unspent$(i) = e(i) - \text{spent}(i)$. For good j, let forced(j) denote the amount of good j sold to all buyers under their forced allocations and let unsold$(j) = 1 - \text{forced}(j)$.

First ensure that for each buyer, i, unspent$(i) \geq 0$ and for each good j, unsold$(j) \geq 0$; otherwise, $\boldsymbol{p}$ do not constitute equilibrium prices. The network $N(\boldsymbol{p})$ is defined over vertices $\{s\} \cup G \cup B \cup \{t\}$, where s and t are its source and sink. For each good j, there is edge (s, j) with capacity unsold$(j) \cdot p_j$ and for each buyer i, there is edge (i, t) with capacity unspent(i). For each buyer i, $N(\boldsymbol{p})$ will contain an edge (j, i) corresponding to each segment s in its flexible partition, Q_{k_i}, where good$(s) = j$; the capacity of this edge is amount$(s) \cdot p_j$.

Lemma 1 *Prices $\boldsymbol{p}$ constitute equilibrium prices iff max-flow in $N(\boldsymbol{p})$ is* $\sum_{i \in B} \text{unspent}(i)$.

3.2 Proofs of Rationality for Fisher and Arrow-Debreu Markets

Let $\boldsymbol{p}'$ be any equilibrium prices for $\mathcal{M}$. Consider all forced allocations made at equilibrium. For each buyer i, let Q_{k_i} denote i's flexible partition in this equilibrium and let L_i denote the set of goods of the segments in Q_{k_i}. Let $R_i = G - L_i$ be the remaining goods. For $j \in L_i$, let s_{ij} denote the segment of good j that is in Q_{k_i}. For $j \in R_i$, let s_{ij} denote the last segment of good j that is fully allocated to i and let s'_{ij} denote the next (unallocated) segment; if no segment of good j is allocated to i, then $s_{ij} = \phi$. Next, we will construct an LP which will have a variable, p_j, corresponding to each good j, and any optimal solution to this LP will be equilibrium prices. The equilibrium $\boldsymbol{p}'$ considered above must be one of its solutions and since the LP has only rational parameters, it must have a rational solution as well, thereby completing the proof.

First write spent(i) and unspent(i) for each buyer as linear polynomials using the variables p_j's. For each good j, unsold(j) is a constant determined by the forced allocations and hence the left-over value of this good, unsold$(j) \cdot p_j$ is a linear expression. Construct the network, say N, described in Section 3.1, except that the capacities of edges will be linear polynomials in the p_j's. We will add edge (t, s) of unbounded capacity to the network. This constitutes the set E of edges of N. Next, we introduce a variable f_e corresponding to each edge e in N, which will represent the flow on this edge.

We can finally describe the LP itself. Its objective is to maximize $f_{(t,s)} + \sum_{i \in B} \text{spent}(i)$, subject to capacity constraints on each edge $e \in E - \{(t, s)\}$ and a flow conservation equation for each vertex in $\{s, t\} \cup G \cup B$. In addition, for each buyer i, it has the following constraints to ensure that the forced and flexible segments of i satisfy desired properties; eventually this ensures that i indeed gets a utility maximizing bundle of goods.

- For each $j, j' \in L_i$, we have the equation: slope$(s_{ij}) \cdot p_{j'} = \text{slope}(s_{ij'}) \cdot p_j$.

• For each $j \in L_i$ and $j' \in R_i$, if $s_{ij'} \neq \phi$, we have the two inequalities: $\text{slope}(s_{ij}) \cdot p_{j'} \geq \text{slope}(s_{ij'}) \cdot p_j$ and $\text{slope}(s_{ij}) \cdot p_{j'} \leq \text{slope}(s'_{ij'}) \cdot p_j$.
• If $s_{ij'} = \phi$, we have one inequality: $\text{slope}(s_{ij}) \cdot p_{j'} \leq \text{slope}(s'_{ij'}) \cdot p_j$.
• Add the following constraints using the linear expressions derived above: $\forall i \in B: \ \text{unspent}(i) \geq 0$, $\forall j \in G: \ \text{unsold}(j) \geq 0$, and $\sum_{j \in G} p_j = M$, where M is the total money of all buyers.
• Finally, we add non-negativity constraints: $\forall e \in E: f_e \geq 0$ and $\forall j \in G: \ p_j \geq 0$.

Clearly, the starting equilibrium prices $\boldsymbol{p'}$ form an optimal solution, of value M, for the LP constructed. One half of Theorem 2 follows from Lemma 1 and the fact that this LP must have an optimal rational solution.

Next we deal with the Arrow-Debreu case [1]. Such a market under piecewise-linear, concave utilities differs from a Fisher market only in that each agent i does not come to the market with money but with an initial allocation $w(i) = (w_{i1}, \ldots, w_{ig})$ of goods; each of the goods still totals 1 unit in the market, w.l.o.g. For any prices of the goods, the agents sell all their initial endowments at these prices and use the money to buy optimal baskets. The problem again is to find market clearing prices.

The main change needed in the proof is that at given prices of goods, $\boldsymbol{p}$, we will let e_i denote the total value of i's initial endowment. If $\boldsymbol{p}$ is a vector of variables, then e_i will be a linear sum $\sum_j w_{ij} p_j$ in these variables. The sum M of the prices can be set arbitrarily (if p is an equilibrium in an Arrow-Debreu market, then αp is also an equilibrium for all $\alpha > 0$), thus we can set w.l.o.g. $M = 1$. The rest of the proof is same as before. Hence we get:

Theorem 2 *Let $\mathcal{M}$ be a Fisher or Arrow-Debreu market with additively-separable piecewise-linear, concave utilities and all parameters rational. If $\mathcal{M}$ has an equilibrium, then it admits an equilibrium in which prices are rational numbers that can be written using polynomially many bits.*

4 Membership in PPAD for Fisher and Arrow-Debreu Markets

Consider the Arrow-Debreu market with a set B of n agents (buyers) $1, \ldots, n$ and a set G of g goods $1, \ldots, g$. Each agent i has a given initial endowment (supply) vector $w(i) = (w_{i1}, \ldots, w_{ig}) \geq 0$ of goods, a given (nonnegative, nondecreasing) concave piecewise-linear utility function f_j^i for each good $j \in G$, and his overall utility is $u_i(x_1, \ldots, x_g) = \sum_{j \in G} f_j^i(x_j)$. We may assume w.l.o.g. that the total initial supply of each good j is equal to 1, i.e., $\sum_i w_{ij} = 1$. We assume for computational purposes that all the input numbers (endowment vectors, slopes and breakpoints of the utility functions) are rationals.

As is well known, a Fisher market F can be reduced to an Arrow-Debreu market D with the same set G of goods, the same set B of agents, and the same utility functions. Assume w.l.o.g. that the total supply of each good in F is 1 and that the sum of the budgets of the agents is also 1. If an agent i has budget e_i in F, then his initial endowment $w(i)$ in D contains the same amount $w_{ij} = e_i$ for each good $j \in G$. Then a price vector p is an equilibrium in F if and only if p is an equilibrium in D. Thus, Fisher markets correspond essentially to the special case of Arrow-Debreu markets, where every agent's endowment contains the same amount of each good.

From the Arrow-Debreu theorem, a sufficient condition for the existence of an equilibrium for an Arrow-Debreu market in our setting is that (C1) all agents i have positive initial endowments w_{ij} for all goods j, and (C2) nonsatiation of the agents' utility functions: for every bundle, there is another bundle that gives strictly more utility to each agent. In our case of piecewise linear functions, (C2) can be equivalently stated as: for every agent $i \in B$, there is a good $j \in G$ such that $\lim_{x \to \infty} f_j^i(x) = \infty$, i.e., the last (infinite) segment of f_j^i has positive slope; we will say that agent i is *nonsatiated* with respect to good j, and the function f_j^i is nonsatiated. Since the initial total supply of each good is assumed to be 1, it suffices actually to assume that each agent derives increasing utility from some good up to an amount greater than 1 (i.e. the utility function f_j^i could go flat after some value > 1).

Some weaker sufficient conditions for the existence of an equilibrium have been shown subsequently by other authors. In particular, Maxfield [20] showed a sufficient condition in terms of the following *economy graph*: The graph has a node for each agent $i \in B$ and has an arc $i \to j$ if there is a good $k \in G$ such that $w_{ik} > 0$ and j is nonsatiated with respect to k. The sufficient condition is: (C') The economy graph is strongly connected. Clearly, (C') implies (C2), and the conjuction of (C1) and (C2) implies (C'). Note that in a Fisher market, each agent has a positive initial budget e_i, and thus, when we express a Fisher market in the Arrow-Debreu framework with an initial endowment $w(i) = (e_i, \ldots, e_i)$, condition (C1) is automatically satisfied; in this case (C2) and (C') are equivalent.

We will show that, under the above sufficient conditions, the problem of computing a (exact) price equilibrium is in PPAD. As part of the proof, we will show also the sufficiency of the conditions for the existence of an equilibrium.

Theorem 3 *The problem of computing a (exact) price equilibrium for an Arrow-Debreu market with additively-separable, piecewise-linear concave utility functions, satisfying the condition (C'), is in PPAD. The same is true for the Fisher market under the condition (C2).*

The proof is rather long and involved and will be provided in the journal version of the paper. We sketch below the main steps. We are given an instance of an Arrow-Debreu market as above, satisfying the sufficient condition (C'). Trim each utility function f^i_j so that it goes flat after 1.1 unit of good j; this does not change the price equilibria. Let S be the unit g-simplex for the prices, $S = \{p | p \geq 0, \sum_j p_j = 1\}$, and let D be the box $[0, 1.1]^{ng}$ of possible demand vectors (allocations) $x = (x_{ij} | i \in B, j \in G)$. Define the correspondence mapping F from $S \times D$ to itself which takes a pair (p, x) and maps it to the set of all pairs (p', x') where: p' is a price vector that maximizes $p'x = \sum_{ij} p'_j x_{ij}$ subject to $p' \in S$, and x' is a demand vector that consists of optimal budget-feasible bundles (in D) for the buyers under prices p. A point (x, p) is a *fixed point* of F if $(p, x) \in F(p, x)$. The price components of the fixed points of F are precisely the price equilibria of the market.

Let b be the maximum number of bits in numerator and denominator of the (rational) input numbers. Assume wlog that the slopes of all nonflat segments of the utilities are integers > 0, and let t be the total number of segments. Let m be the number of bits that suffice in the optimal solution of LPs with at most $3ng$ variables, $t^2 + 5ng$ constraints, and rational coefficients of bit complexity $2b$. Note that m is polynomially bounded in n, g, t, b, and $m >> n, g, t, b$. Let $\delta = 1/2^{10m}$.

Consider a regular simplicization of the domain $S \times D$ with resolution δ. Every cell (little simplex) in the simplicization has rational vertices which are equal in each coordinate or differ by δ. Define a function G which picks at each vertex (p, x) of the simplicization an arbitrary element of $F(p, x)$, and is extended to the domain $S \times D$ by linear interpolation. We'll denote by $G1, G2$ the $p-$ and $x-$ components of G. By definition, G is a continuous, piecewise linear function and is it easy to see that is polynomial-time computable. Thus, computing a (exact) fixed point (p^*, x^*) of it is in PPAD [13]. Let C be the simplex that contains (p^*, x^*). Note, a fixed point of G is not a fixed point of F. We show that (p^*, x^*) has a sequence of properties, which allow us eventually to compute a price equilibrium.

Lemma 4 *The total demand for each good j in x^* is $\sum_i x^*_{ij} \leq 1 + 4ng\delta$, i.e. approximately bounded by the supply.*

Lemma 5 *For each agent i, $\sum_j p^*_j x^*_{ij} \leq \sum_j p^*_j w_{ij} + 2.2g\delta$, i.e. (p^*, x^*) is "almost" budget-feasible for each agent. Also, $\sum_j p^*_j x^*_{ij} \geq \sum_j p^*_j w_{ij} - 2.2g\delta$ and $\sum_{i,j} p^*_j x^*_{ij} \geq 1 - 2.2g\delta$.*

Consider the utility function f^i_j of agent i for good j, and the l-th segment of the function; let s_{ijl} be its slope and suppose the segment runs from amount c_{ijl} to $c_{ij,l+1}$ for the good j. For a demand vector x, we say that the segment is *empty* (resp. *full*) if $x_{ij} \leq c_{ijl}$ (resp. $\geq c_{ij,l+1}$); we say it is *partial* if x_{ij} is between the two amounts. For each good j, there is a last segment which is full and a first segment which is empty; either there is a partial segment which is between the two - we call this the *active* segment - or the two segments are consecutive and the amount x_{ij} is the common breakpoint. Let us say that a segment is *almost full* if it is full to a fraction $> 1 - 2^{-2m}$ of its length and *almost empty* if it has $< 2^{-2m}$ fraction of its length. Condition (C') (or C1) is needed for the following lemma.

Lemma 6 *1. All agents have budget (income) at least $1/2^m$ at p^*.*
2. Suppose that $p^*_j < 2^{-3m}$ for some good $j \in G$. Then all the segments of good j that have positive slope are full in x^*.

We show now that the allocation x^* is approximately consistent with the bang-per-buck order of all the segments in the utility functions of every buyer with respect to the prices p^*. Recall that t is the total number of segments of the utility functions.

Lemma 7 *The following holds for the demand vector x^* for each buyer i and each pair of goods $j1, j2$. If $l1$ is a full or partial segment of $j1$ and $l2$ is an empty or partial segment of $j2$, both with positive slopes, then the slopes of the segments and the vector p^* satisfy $p^*_{j1}/s_{i,j1,l1} \leq p^*_{j2}/s_{i,j2,l2} + 2t\delta$, unless both $l1, l2$ are partial and $l1$ is almost empty and $l2$ is almost full.*

Assume we have a fixed point (p^*, x^*) of the function G. Compute for each buyer the full, partial, and empty segments wrt x^*. We will set up a Linear Program, whose solution will give us a (exact) price equilibrium. The variables of the LP are the same as for proving rationality, i.e. prices p_j, flows f_{ij} for buyer i, good j , corresponding to the costs of the allocations on the active segments , and in addition variable ϵ for the error (tolerance). The LP is: minimize ϵ subject to a set of constraints. For every pair of segments $(i, j1, l1), (i, j2, l2)$ of the same buyer i, if their slopes and the vector p^* satisfy $p^*_{j1}/s_{i,j1,l1} \leq p^*_{j2}/s_{i,j2,l2} + 2t\delta$, then we include a constraint $p_{j1}/s_{i,j1,l1} \leq p_{j2}/s_{i,j2,l2} + \epsilon$. For every buyer i and good j, let a_{ij} be the sum of the lengths of all full segments of good j wrt x^*. We have constraints $\sum_j a_{ij}p_j \leq \sum_j w_{ij}p_j + \epsilon$, for all buyers i. We set up the network as in the rationality proof, except that we add ϵ to all the capacities. If a segment is partial but almost empty, then we include the corresponding edge in the network with capacity ϵ. We have flow conservation constraints and capacity constraints. In addition we have constraints that say that the total flow out of s (or into t) is at least $1-\sum_{i,j} p_j a_{ij} - \epsilon$ (i.e. Walras law is almost satisfied), And finally $\sum p_j = 1$, and all variables are ≥ 0.

Using the previous lemmas, it is easy to verify that the vector with $p = p^*$, and flow f = cost of active segments according to x^* and p^*, and $\epsilon = 2^{-2m}$ satisfies all the constraints. The value of this solution is 2^{-2m}. The LP has less than $3nk$ variables, $t^2 + 5ng$ constraints, and rational coefficients of bit complexity $2b$. Thus, there is an optimal solution with bit complexity m, hence the optimal value is either 0 or at least 2^{-m}. Therefore, it is 0. Consider an optimal solution $(\pi, \phi, 0)$. The following lemma completes the proof of the theorem.

Lemma 8 *π is a price equilibrium.*

5 PPAD-hardness for Fisher Markets

Theorem 9 *Computing a price equilibrium of a Fisher market with additively-separable piecewise-linear concave utilities that satisfies condition (C2) is PPAD-hard, and hence PPAD-complete. The computation of a ϵ-approximate equilibrium for $\epsilon = O(n^{-13})$ is also PPAD-complete.*

Our reduction builds on the construction of [3] which proves the PPAD-hardness for Arrow-Debreu markets of computing a ϵ-approximate price equilibrium, i.e. a price vector p for which there is an allocation x that gives each agent an optimal bundle with respect to p, and the market clears approximately in the sense that $|\sum_i x_{ij} - \sum_i w_{ij}| \leq \epsilon \sum_i w_{ij}$ for every good j. Their reduction constructs from a given 2-player game Γ, with n pure strategies for each player, an Arrow-Debreu market D such that a n^{-13}-approximate equilibrium of D can be efficiently mapped to a n^{-6}-well supported approximate Nash equilibrium of the game Γ; the latter problem is PPAD-complete [4]. The constructed market instance D has a set G of $g = 2n + 2$ goods. and two sets B_0, B_1 of agents. The first set B_0 has $g(g-1)$ agents whose definition (endowments and utilities) do not depend on the game Γ; these agents have the bulk of the endowment in the market (each one of them has a supply of $1/n$ units of a good) and serve a "price-regulating" role, ensuring that in every (approximate) equilibrium all the prices are within a factor 2 of each other. The second set B_1 has $2n^2$ agents with much smaller endowment (total $O(1/n^4)$ for each), and their definition encodes the payoff matrices of the game.

Our reduction consists of a simpler gadget (in place of B_0) for price regulation, and essentially a reduction from Arrow-Debreu to Fisher for the rest of the market. We construct a Fisher instance F that has the same set G of $g = 2n+2$ goods. The set of agents consists of a single agent 0 for the price regulation and the same remaining set B_1 of $2n^2$ agents as in the Arrow-Debreu instance D. The budget e_0 of agent 0 is $2 + \frac{1}{n}$, and his utility function for every good j has slope 2 until e_0 units and slope 1 from then on. Every agent k in B_1 is given budget e_k in instance F equal to the maximum amount $\max_j w_{kj}$ of any good in his endowment in instance D; thus all agents in B_1 have budget at most $O(1/n^4)$.

The utility function of an agent $k \in B_1$ in the Fisher market F is defined as follows. Let u^k_j be the utility function in D for each good $j \in G$, and let s_k be the maximum slope of any segment in these functions over all $j \in G$. The utility function f^k_j for good j in the Fisher instance F has slope $3s_k$ until $e_k - w_{kj}$, and from that point on, the additional utility is a copy of the function u^k_j. That is, $f^k_j(x) = 3s_k x$ if $x \leq e_k - w_{kj}$, and $f^k_j(x) = 3s_k \cdot (e_k - w_{kj}) + u^k_j(x - (e_k - w_{kj}))$ if $x > e_k - w_{kj}$.

Let M be the sum of all the budgets; note that the total budget of the set B_1 of agents is $\leq 2n^2 \cdot O(n^{-4}) = O(n^{-2})$, while the budget of agent 0 is $2 + n^{-1}$; thus $M = 2 + n^{-1} + O(n^{-2})$. The total supply of each good is set equal to M. This concludes the definition of the Fisher instance F.

Since there are M units of each good and a total budget of M, the sum of the prices of an equilibrium must satisfy $\sum_j p_j = 1$. We say that p is an ϵ-approximate equilibrium for the Fisher market if $\sum_j p_j = 1$ and there is an allocation x that consists of optimal bundles for all the agents with respect to p (subject to their budgets) such that $|\sum_{i \in B} x_{ij} - M| \leq \epsilon M$ for all goods $j \in G$.

Lemma 10 *In any 0.9-approximate price equilibrium for the above Fisher market instance F, the prices of* *all the goods are positive and are within a factor 2 of each other.*

The heart of the proof of correctness of the reduction is summarized in the following lemma; Theorem 9 follows.

Lemma 11 *Let p be a n^{-13}-approximate equilibrium for the Fisher market F, and x be an allocation that witnesses this. Then the allocation x of the Fisher instance F can be mapped to an allocation y for the Arrow-Debreu instance D that satisfies the conditions witnessing that p is also a n^{-13}-approximate equilibrium for the instance D.*

6 NP-completeness of Existence of Equilibrium

Theorem 12 *The problem of determining whether a given Fisher or Arrow-Debreu market with additively-separable piecewise linear concave utilities has an equilibrium is NP-complete. The same holds for the existence of a ϵ-approximate equilibrium with $\epsilon = O(n^{-5})$.*

Membership in NP follows from the analysis of Sections 4 and 5 and Theorem 2. For the NP-hardness, we reduce from the *Exact Cover by 3-Sets* (X3C) problem [16]. In this problem, we are given a family $\mathcal{C}$ of n sets $C_1, \ldots, C_n$, where each set C_i is a 3-element subset of a set $X = \{x_1, \ldots, x_n\}$. The question is whether there exists a subfamily $\mathcal{C}'$ of $\mathcal{C}$ which covers X exactly, i.e. every element $x_j \in X$ belongs to exactly one set in $\mathcal{C}'$; such a subfamily is called an *exact cover*.

Given an instance of the X3C problem, we construct an instance D of an Arrow-Debreu market and a corresponding instance F of a Fisher market such that the X3C instance has a solution iff D and F have an equilibrium iff they have an approximate equilibrium. The piecewise linearity of the utility functions allows us to encode an X3C instance by the markets in such a way so that certain goods corresponding to the sets must have in an equilibrium one of two possible prices, corresponding thus to a binary choice for the sets, and a combination of prices is part of an equilibrium iff the sets corresponding to one price form an exact cover. The details of the constructions and the proofs will be given in the journal verison of the paper.

7 Discussion

Nash equilibria and market equilibria play a central role in game theory and economics. In the case of games, 2-player games have rational Nash equilibria and the complexity of computing them is characterized exactly by the class PPAD, as shown by two fundamental results, the classical Lemke-Howson algorithm [19] for membership and the reductions of [4, 8] for hardness.

In the case of markets, the class of separable, piecewise-linear, concave utility functions are an important, broad class which, as we showed, have rational equilibria, if any. As we saw, there is no effeciently checkable necessary and sufficient condition for the existence of equilibria for this case, unlike the linear case. However, under standard (mild) sufficient conditions, the results of the present paper together with [3, 7] show that the equilibrium computation problem for this case, for both market models, is characterized exactly by the class PPAD.

3-player games have irrational Nash equilibria in general and the complexity of computing or approximating them is characterized by the class FIXP. Leontief and non-separable piecewise-linear concave utilities also have irrational equilibria in general (under standard sufficient conditions). Are they FIXP-complete?

The definition of the class PPAD was designed to capture problems that allow for path following algorithms, in the style of the algorithms of Lemke-Howson [19] and Scarf [24]. Our result, showing membership in PPAD for both market models under separable, piecewise-linear, concave utility functions, establishes the existence of such path following algorithms for finding equilibria for these market models (and one can obviously derive such algorithms indirectly by the membership proof). It will be interesting to obtain natural direct algorithms for this task (hence leading to a more direct proof of membership in PPAD), which may be useful to compute equilibria in practice.

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