Analytical Tools for Natural Algorithms

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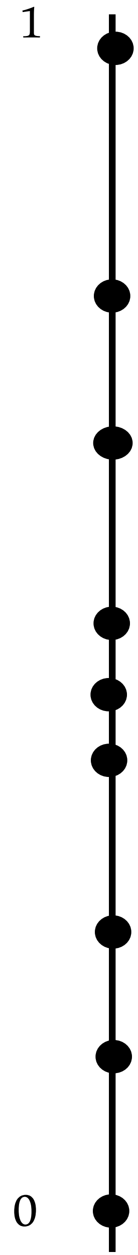
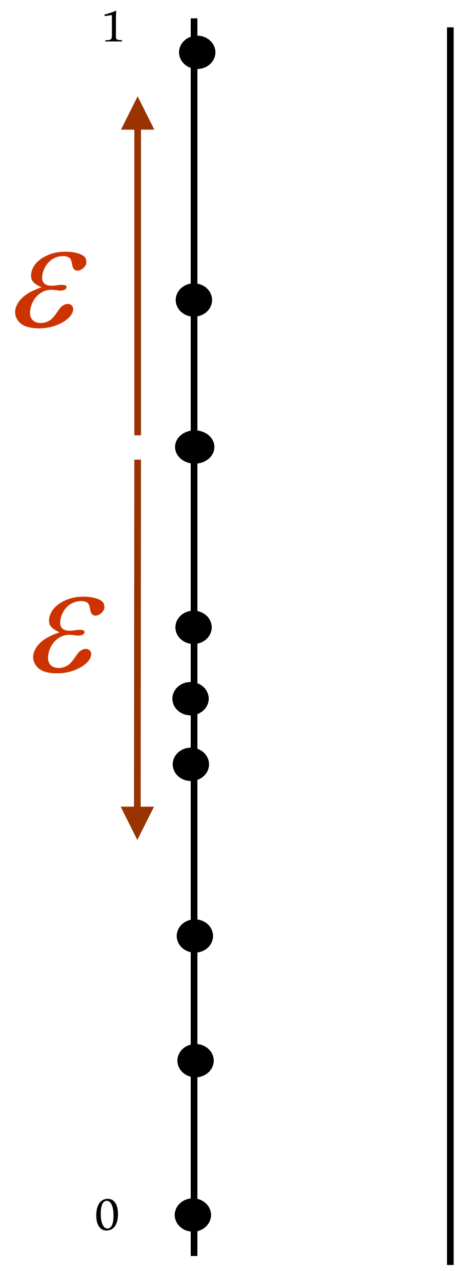
three multiagent agreement systems

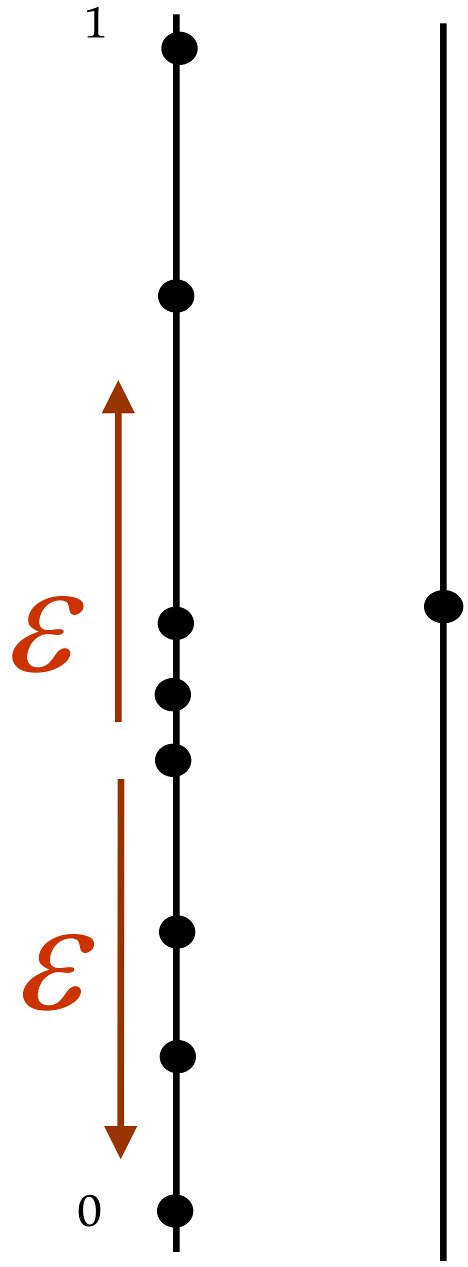


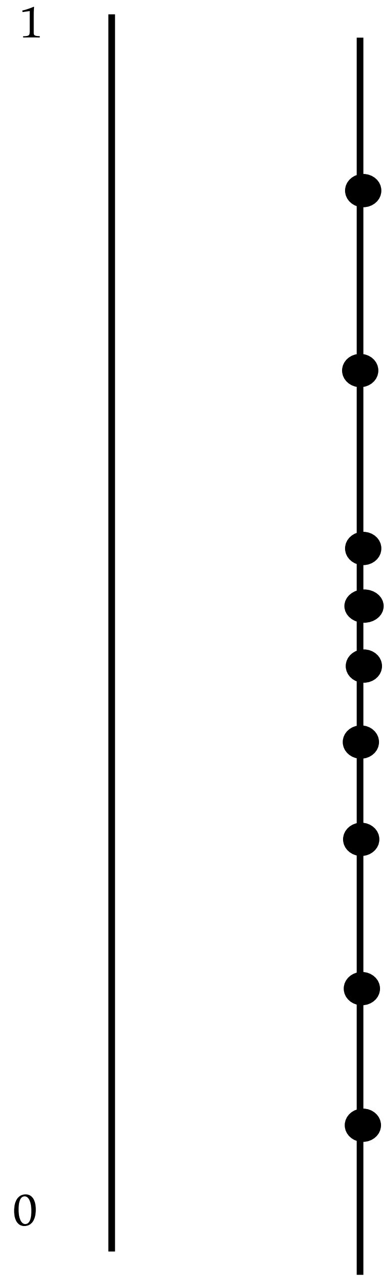
n opinions

Hegselmann-Krause opinion dynamics



each opinion
averages itself
with similar opinions


1
ε
ε
0


1
0

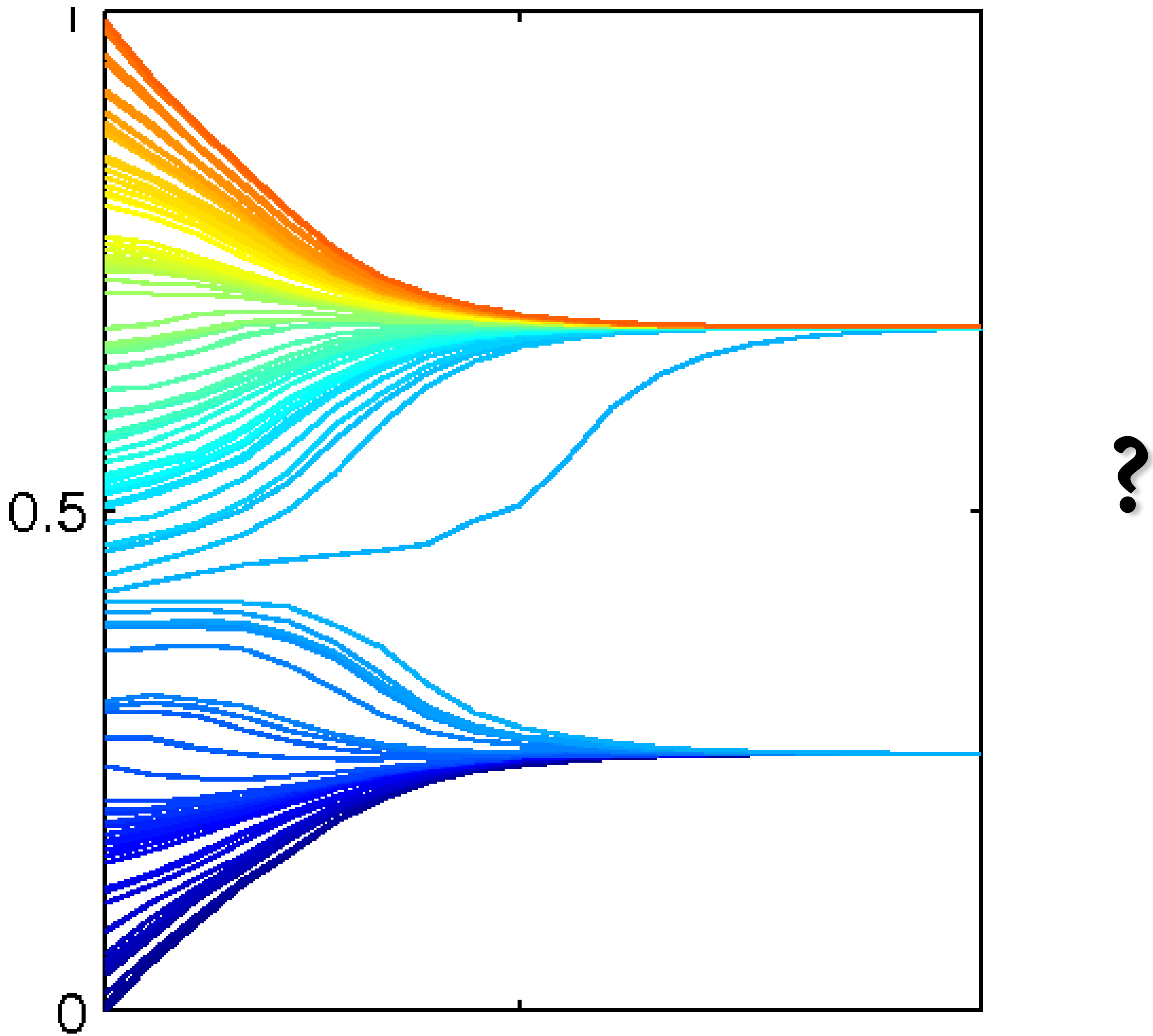


figure by Urbig-Lorenz-Herzberg

Kuramoto sync

theorem

all of these dynamical systems converge in time

$$2^{O(n)}$$

where n is the number of opinions, metronomes, fireflies, etc.

no previous convergence bound was known

let’s step back a little

$$P^t x$$

matrix

vector

$$P^t x$$

predict behavior

for large t

$$P^t x$$

spectral decomposition

$\Downarrow$

dimension separation

$\Downarrow$

independent 1-dim systems

that was was easy

what about?

$$P_t \cdots P_2 P_1 X$$

where

$$P_t = f(x, P_1, \cdots, P_{t-1})$$

nonlinear dynamics

$P_t \cdots P_2 P_1$ x

hopeless

attack

$$P_t \cdots P_2 P_1 \, x$$

as

$$P_t(\cdots(P_2(P_1 x)))$$

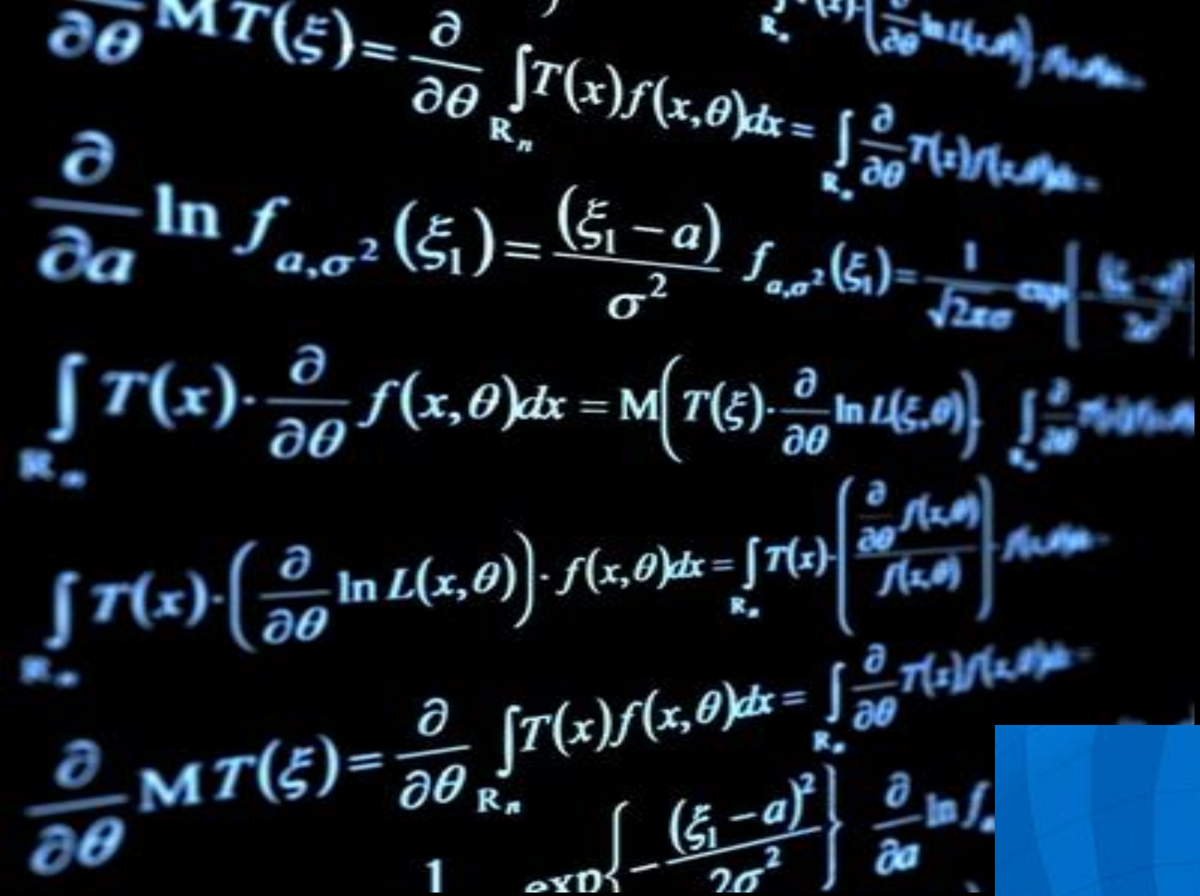
bye-bye old math

hello data analysis

classification

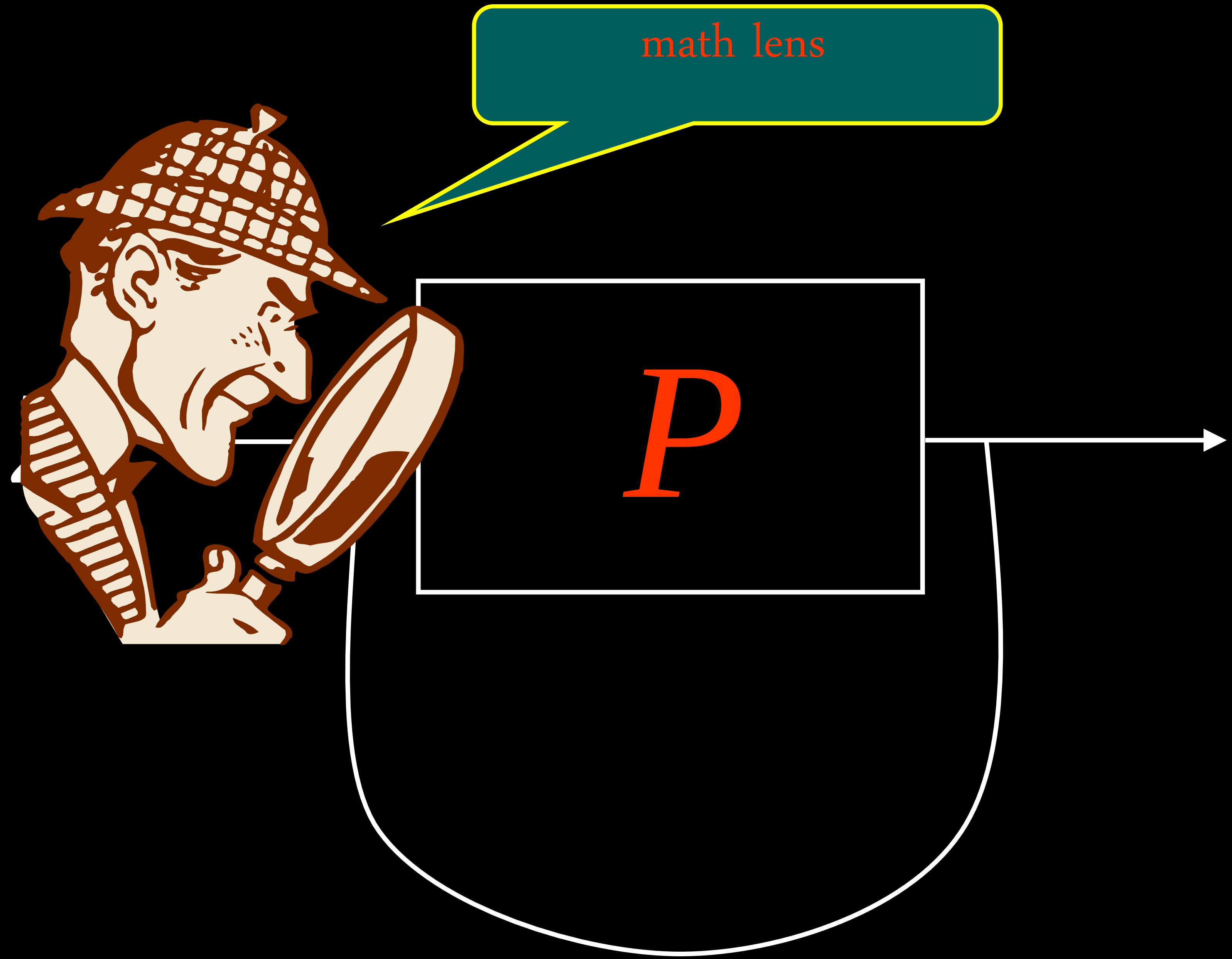
machine learning

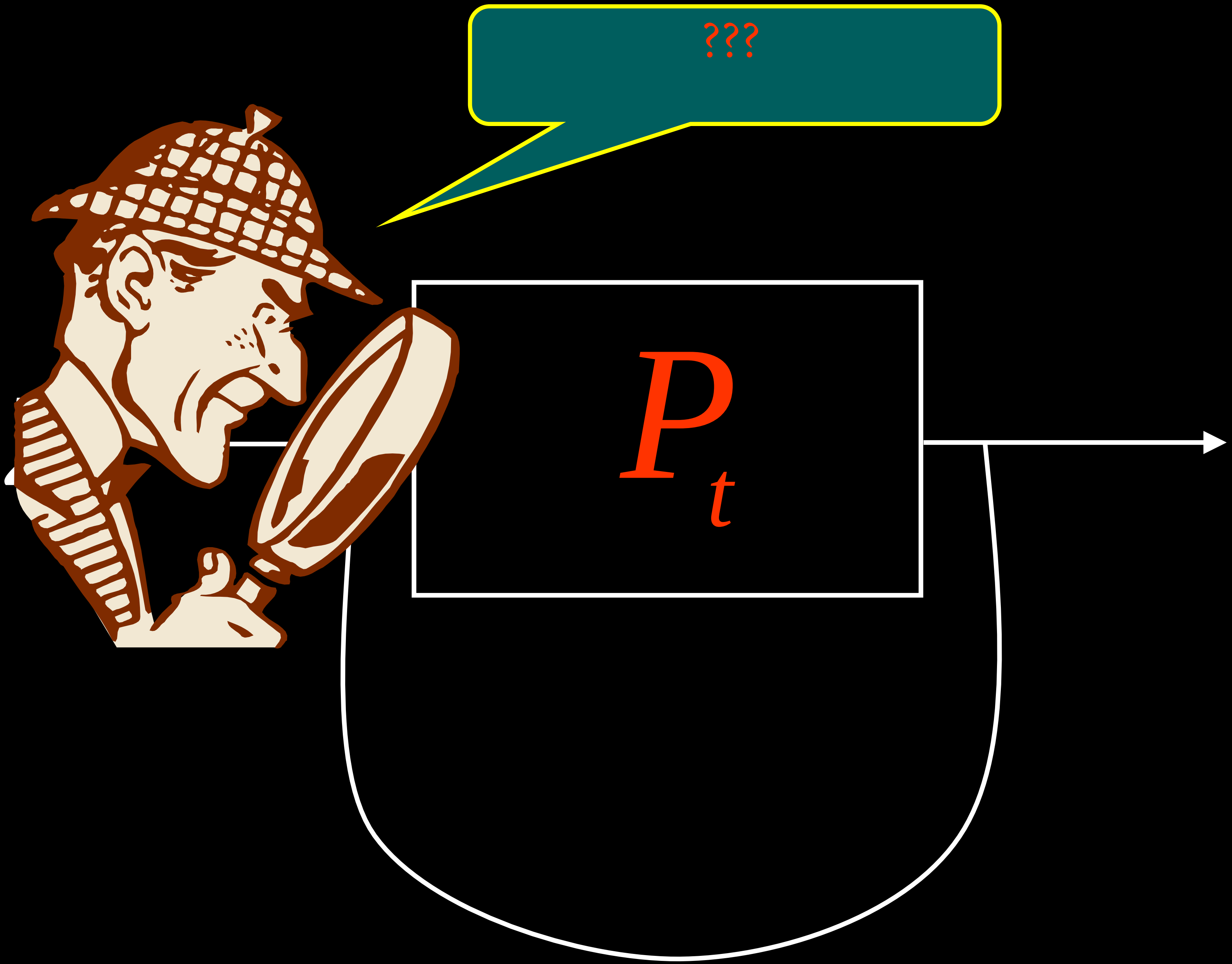
statistics

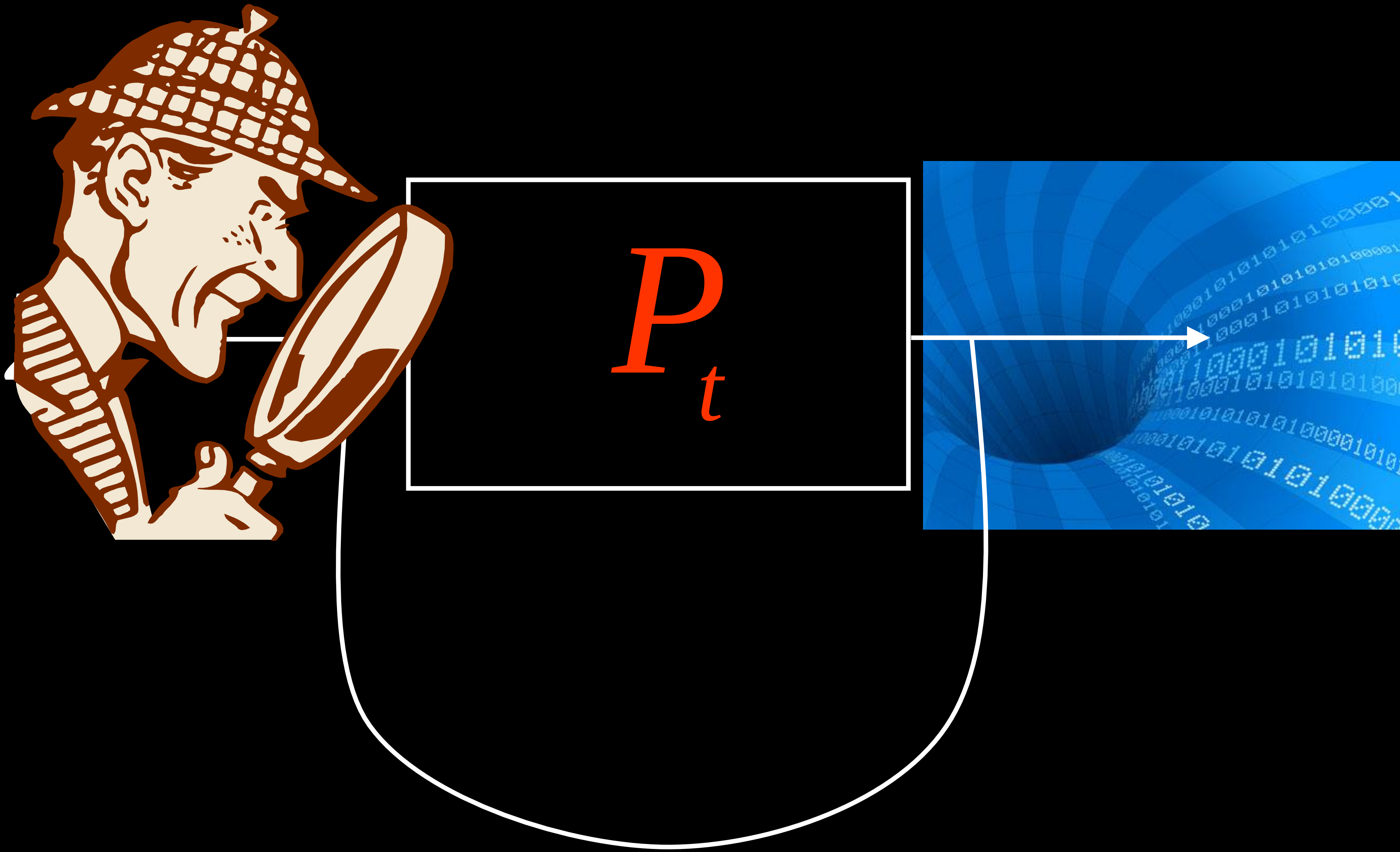


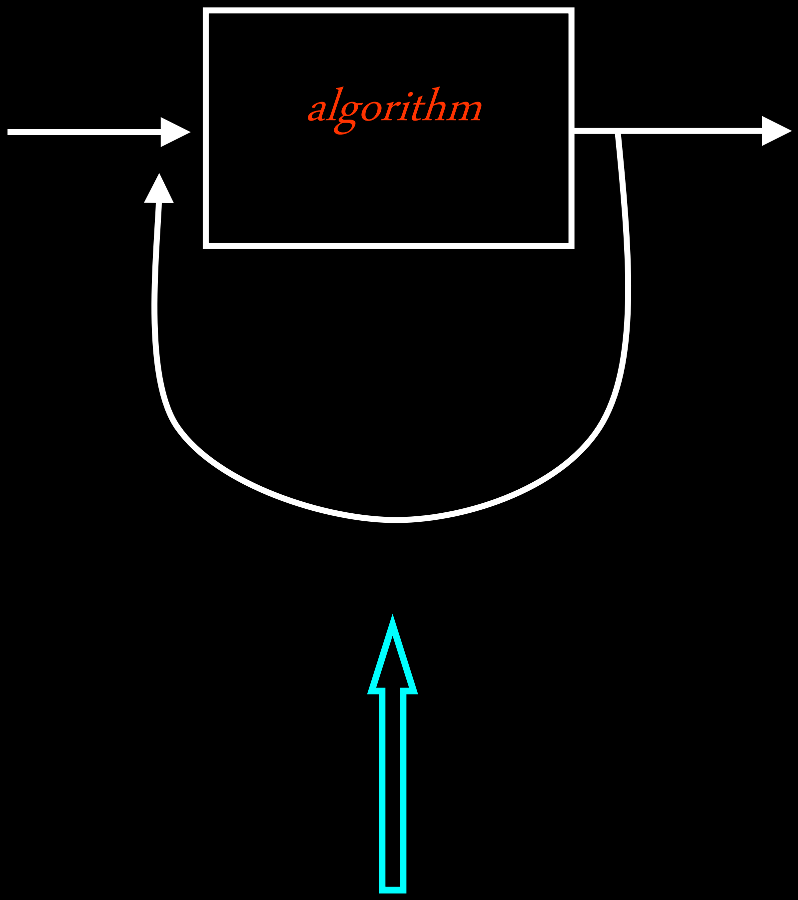
$$\Leftarrow \quad P^t x$$

$$P_t \cdots P_2 P_1 x \quad \Rightarrow$$


math lens
P

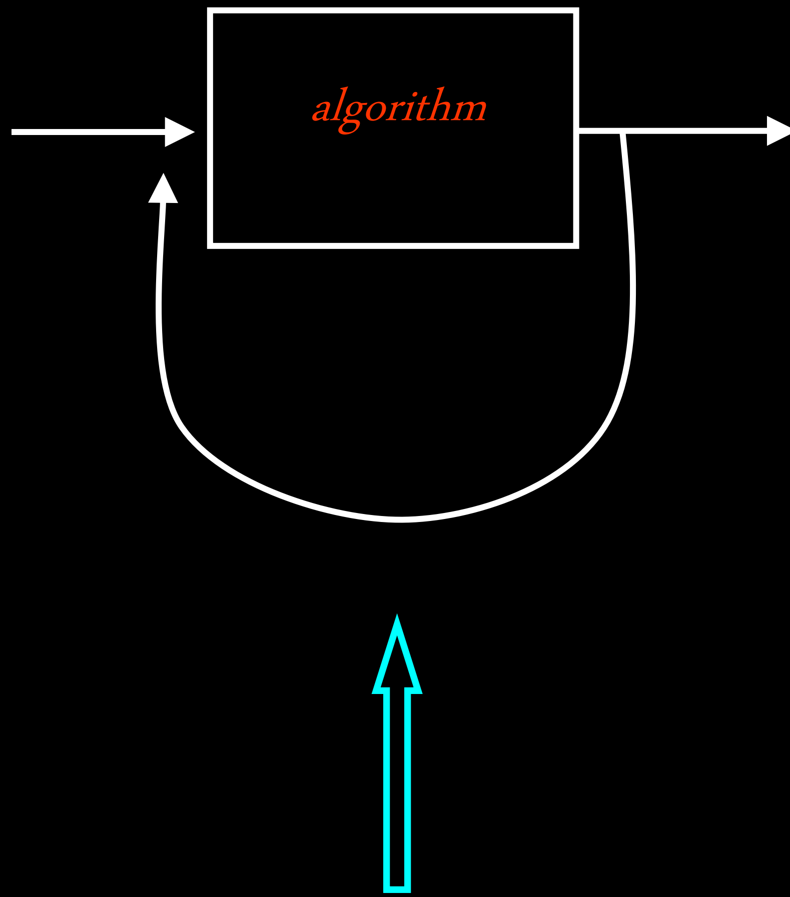

???
P_t


P_t



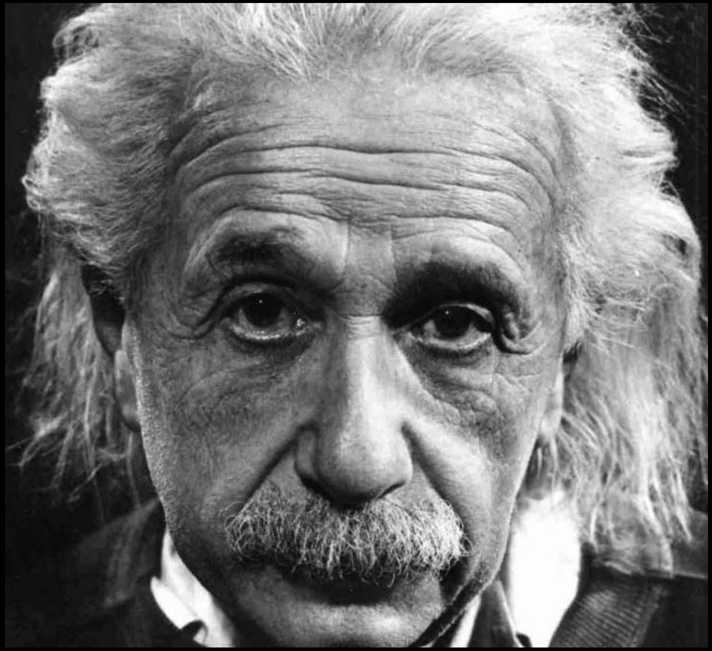
good old math

rarely works

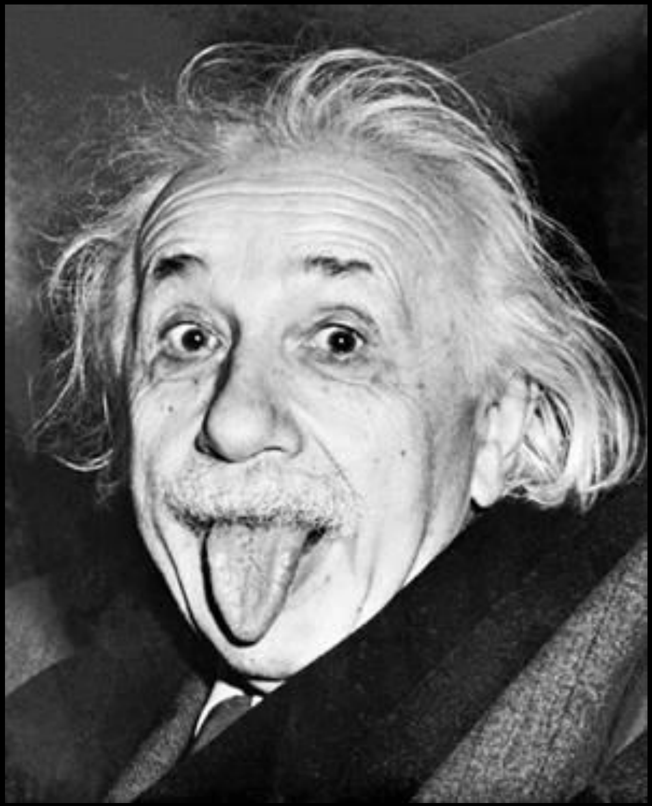


data analysis

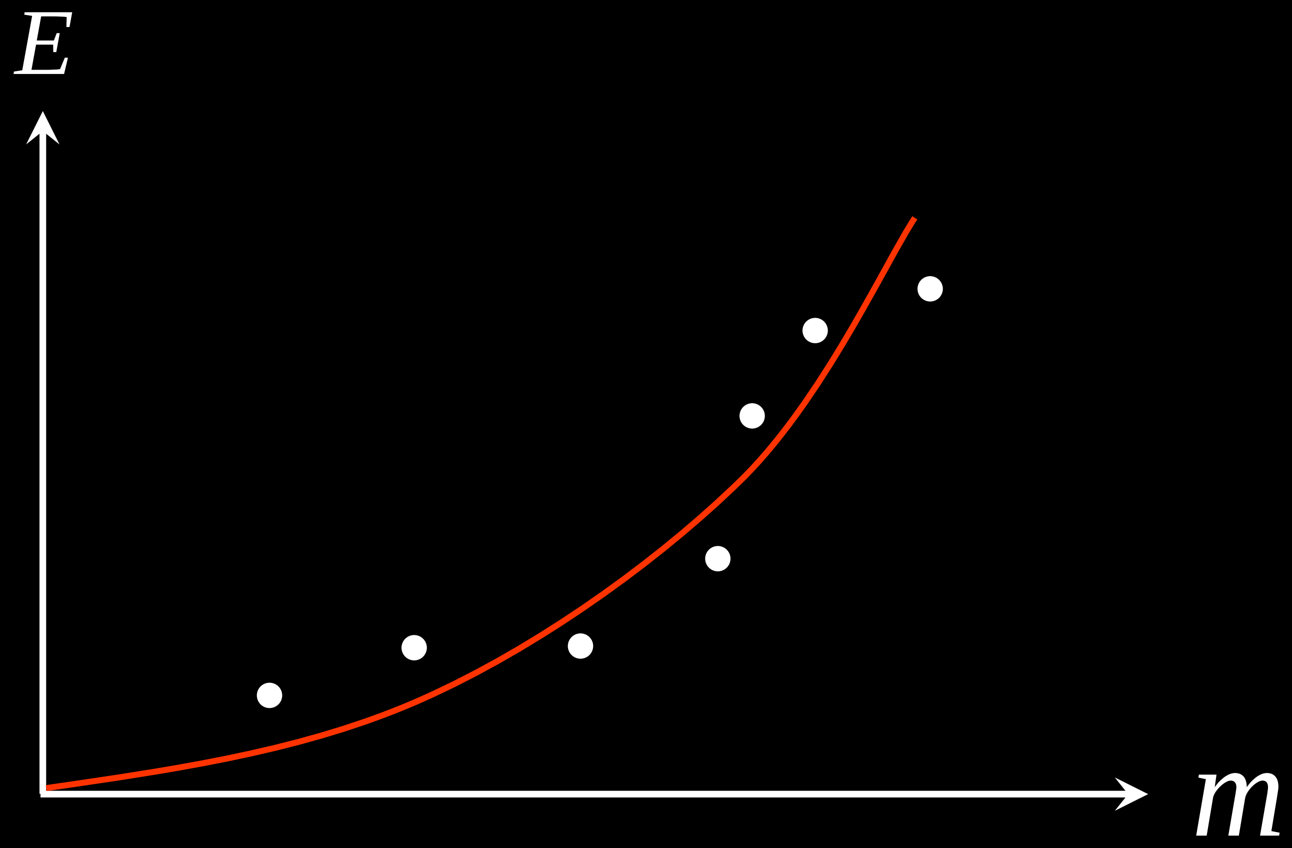
works if you redefine the meaning of “works” relativity theory



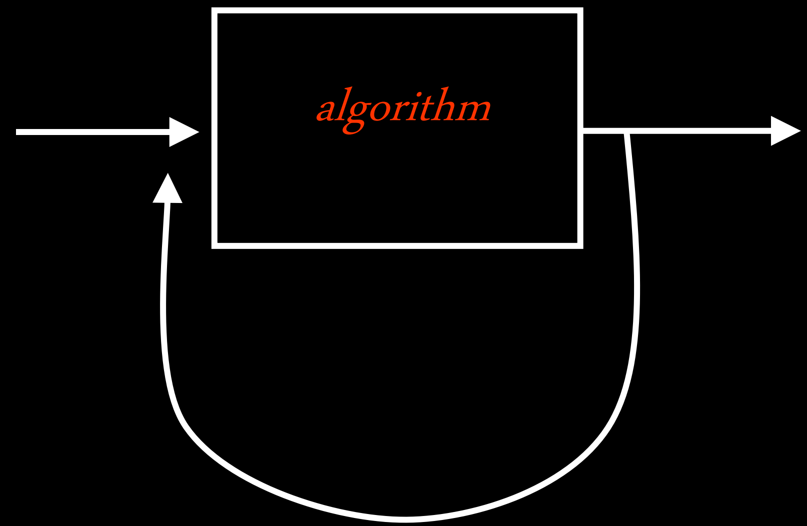
relativity theory via data analysis



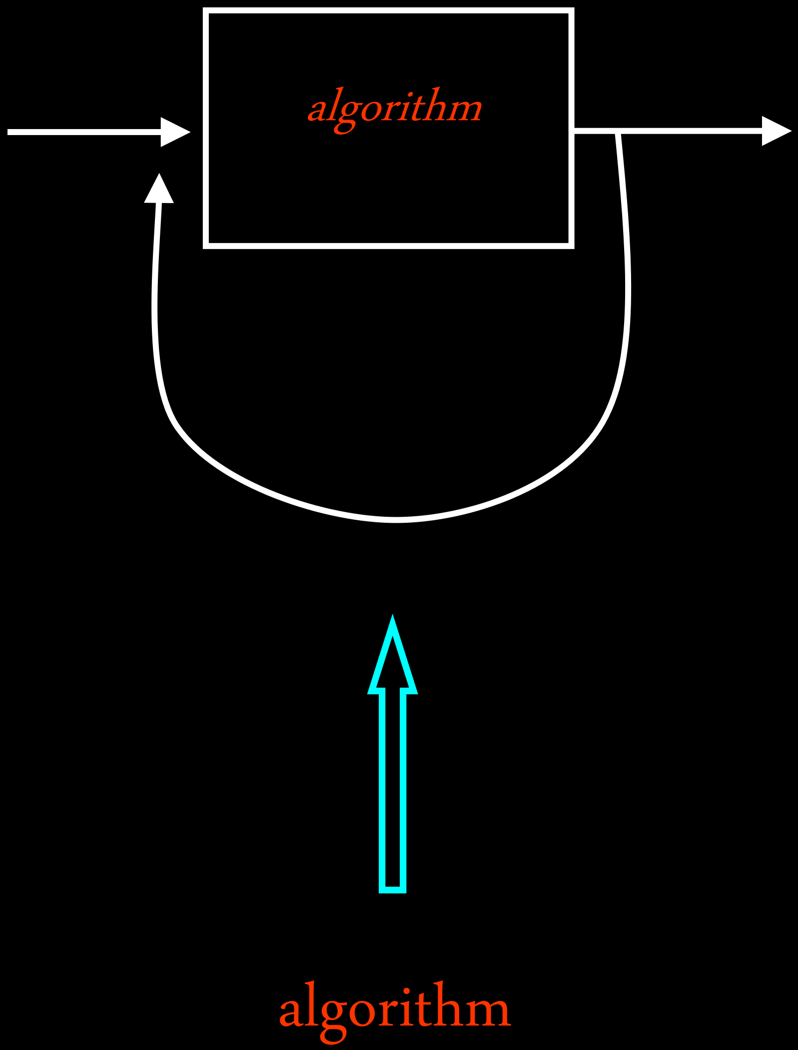
relativity theory via data analysis



$$E = m^{0.92} c^{2.03}$$

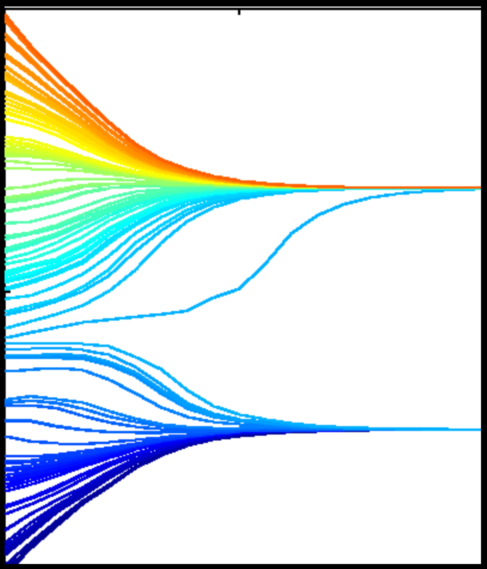


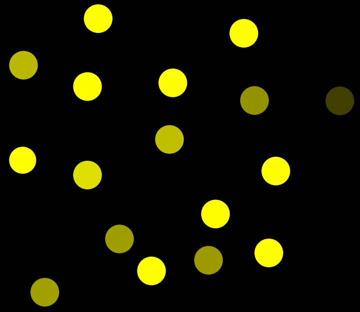
what to do?


algorithm
algorithm

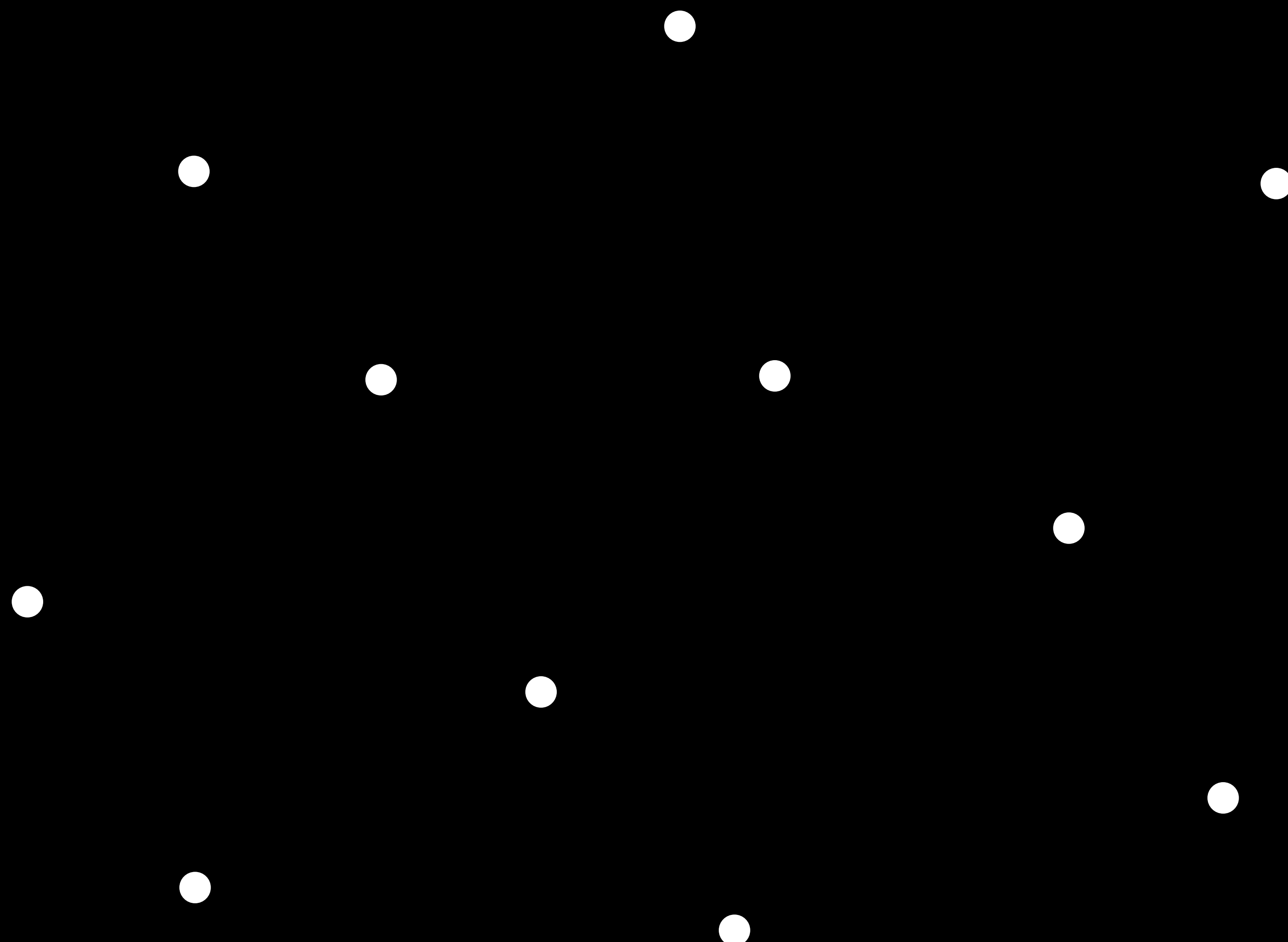
use algorithms to analyze algorithms

back to our 3 agreement systems

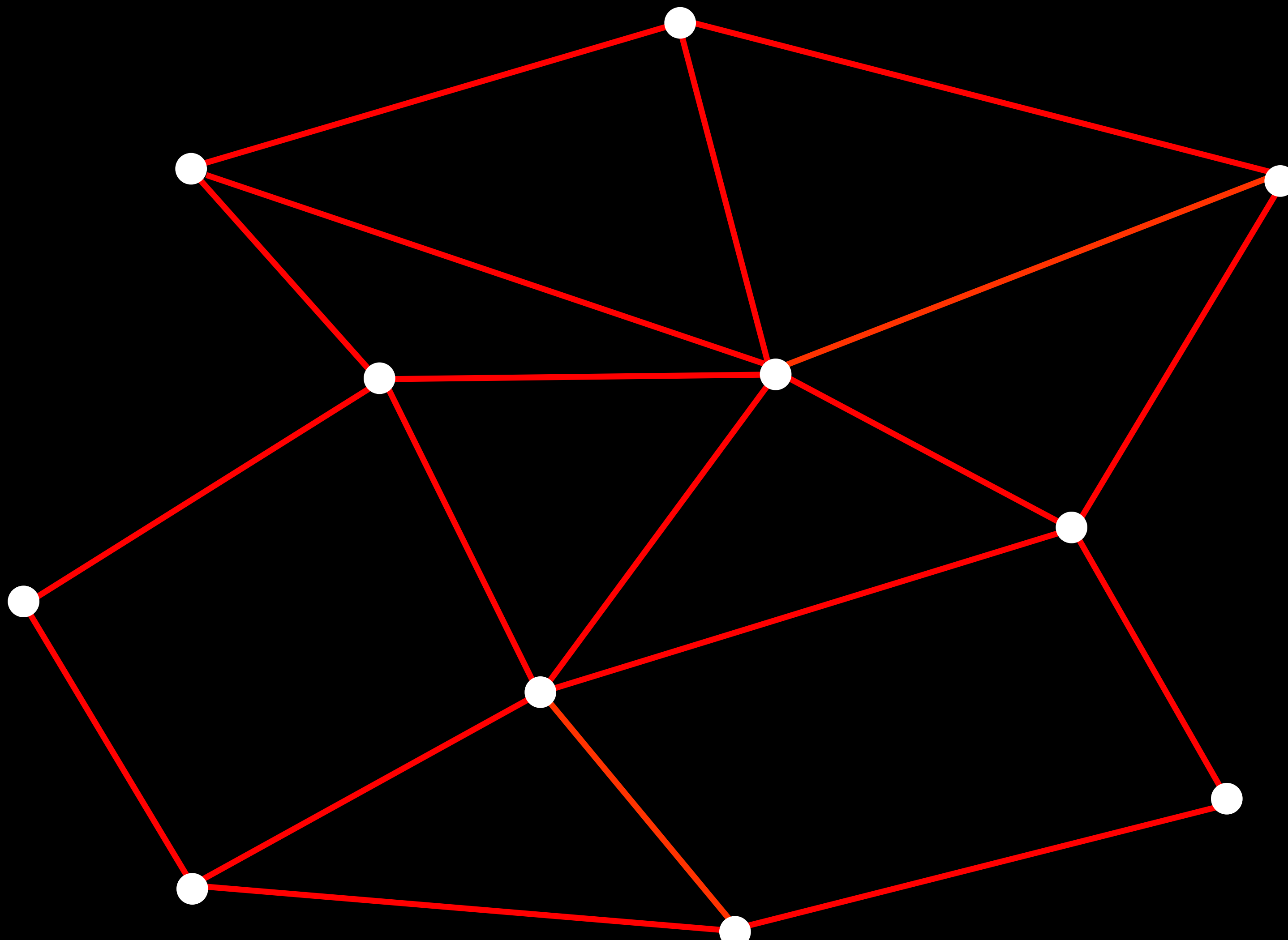




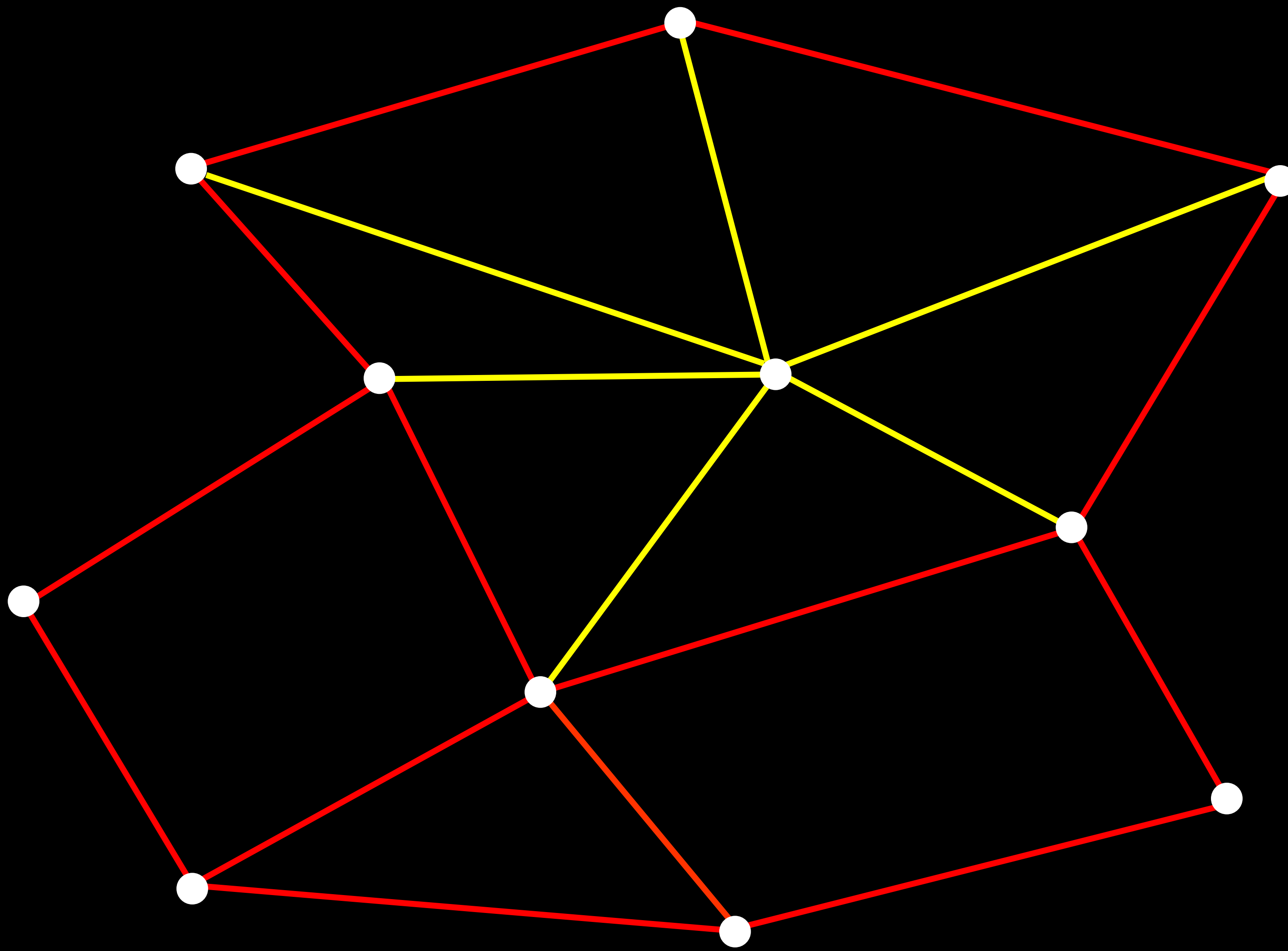
a common geometric framework

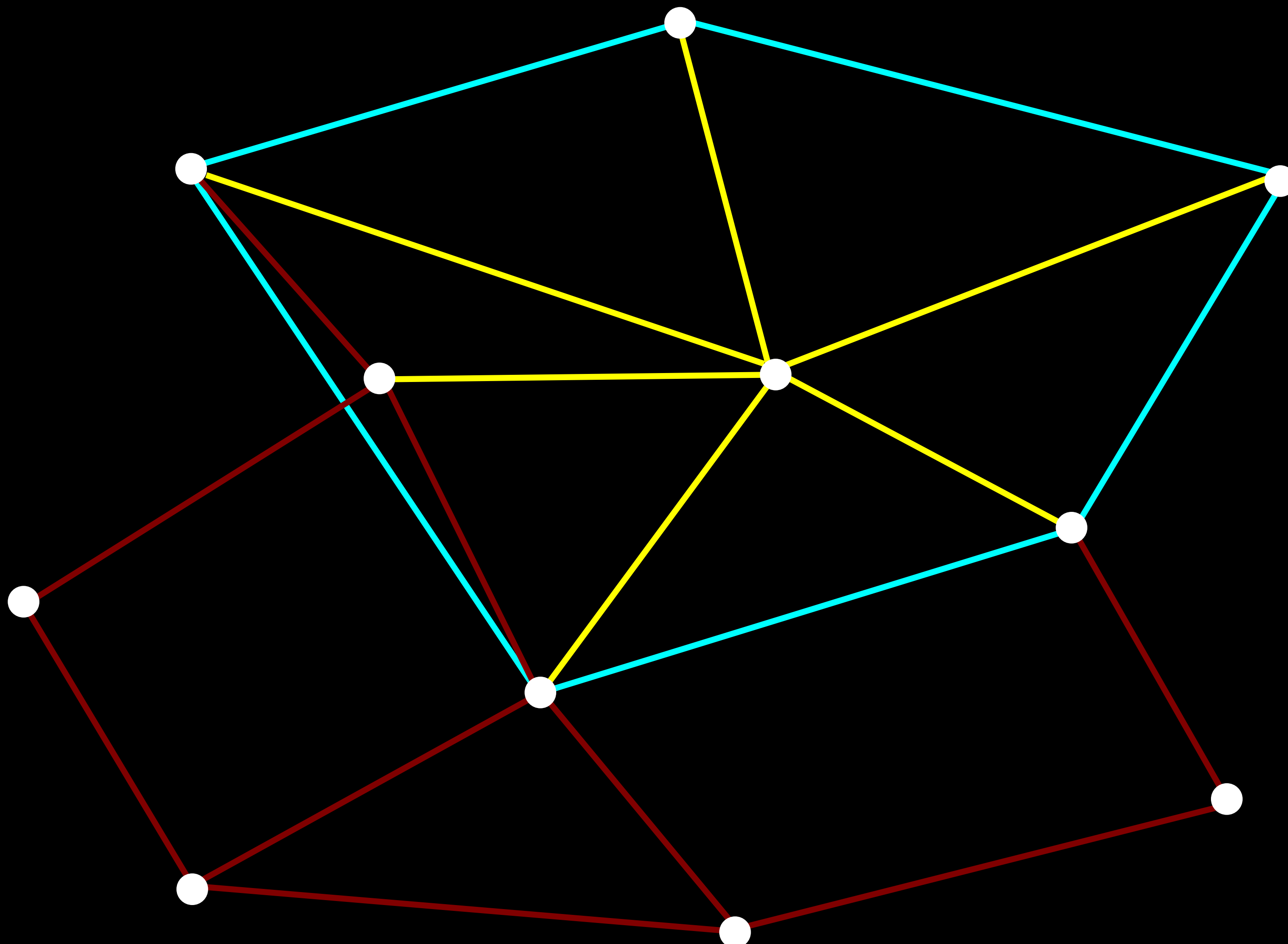


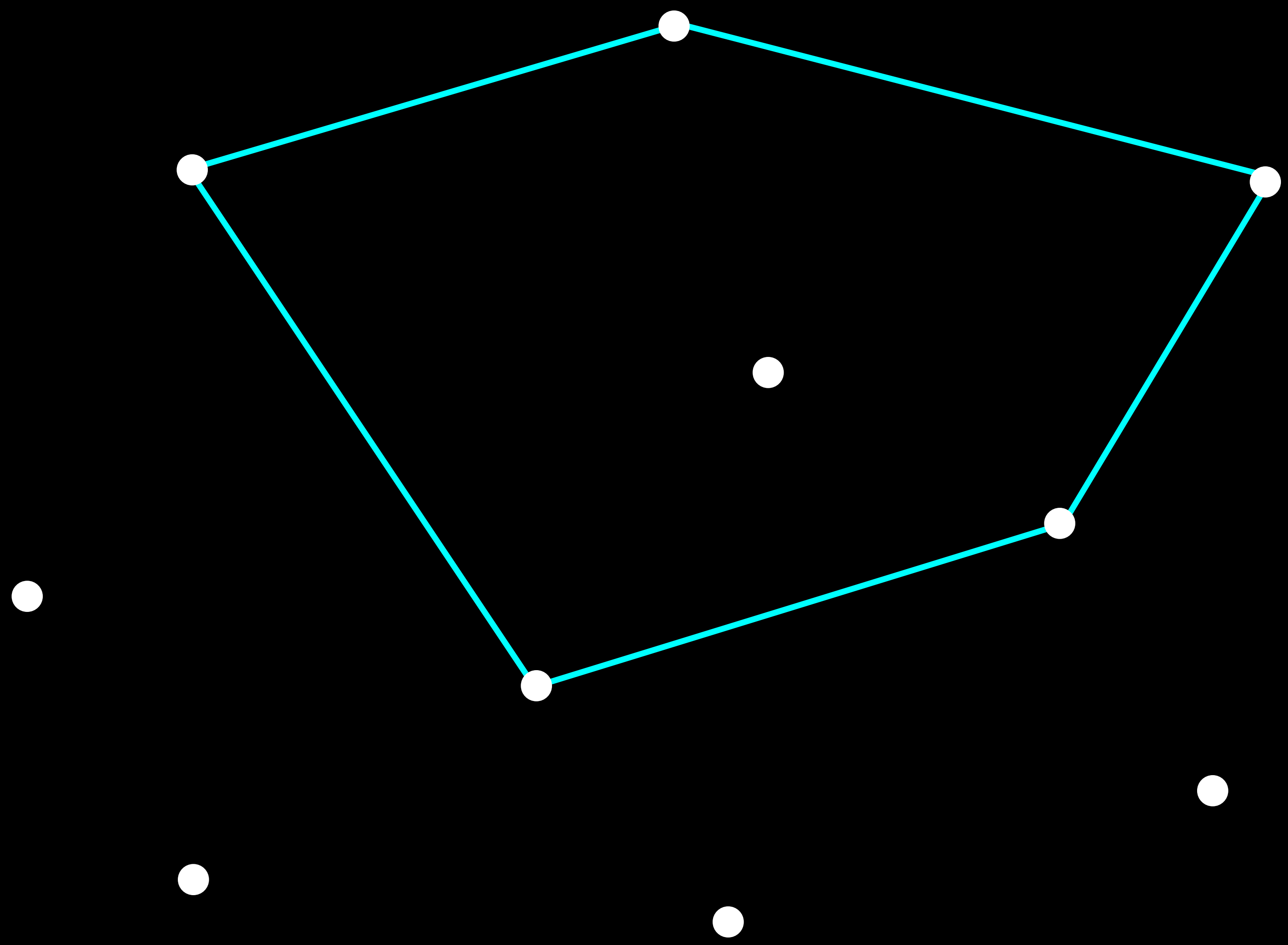
n points in E^d

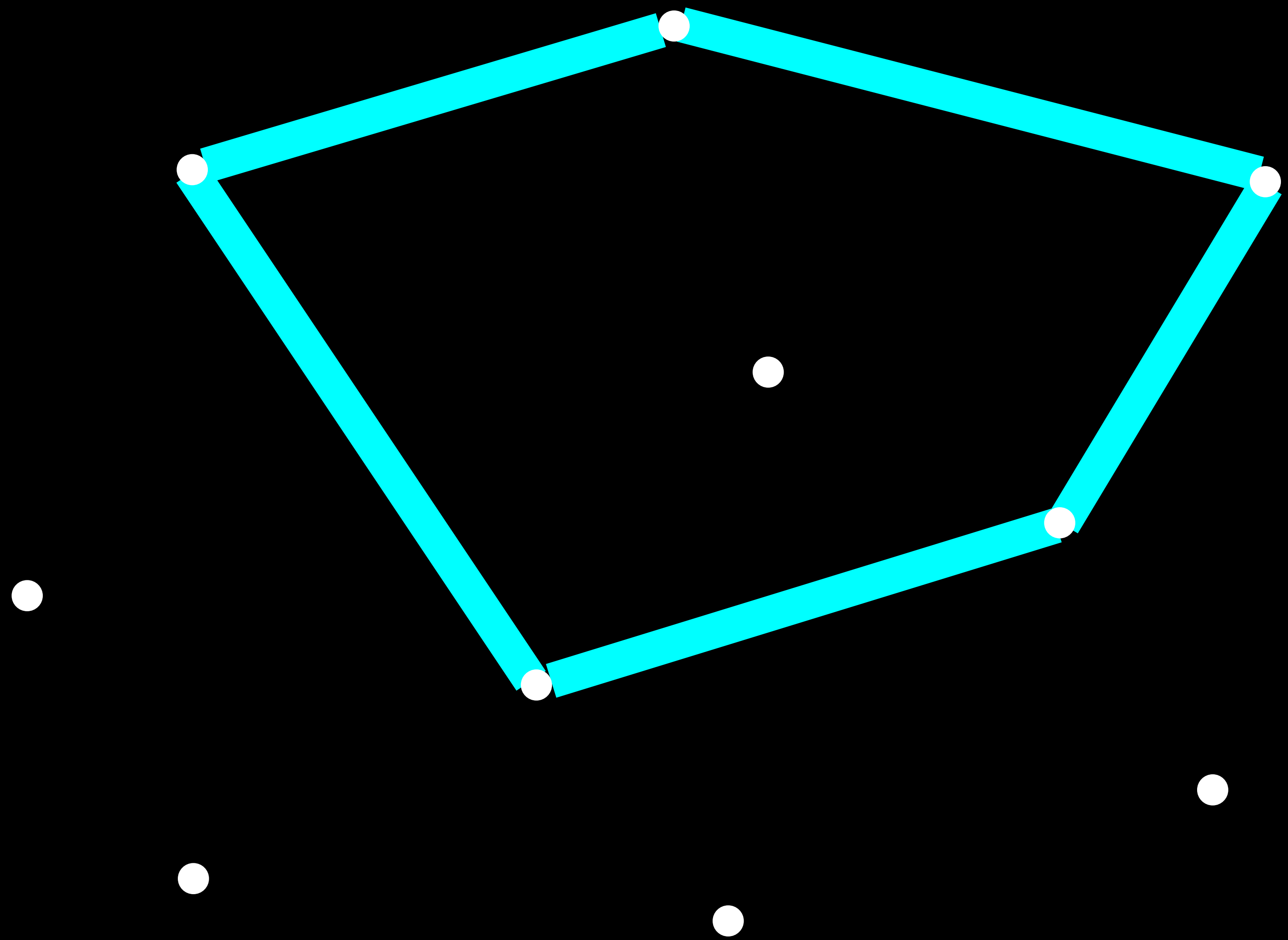


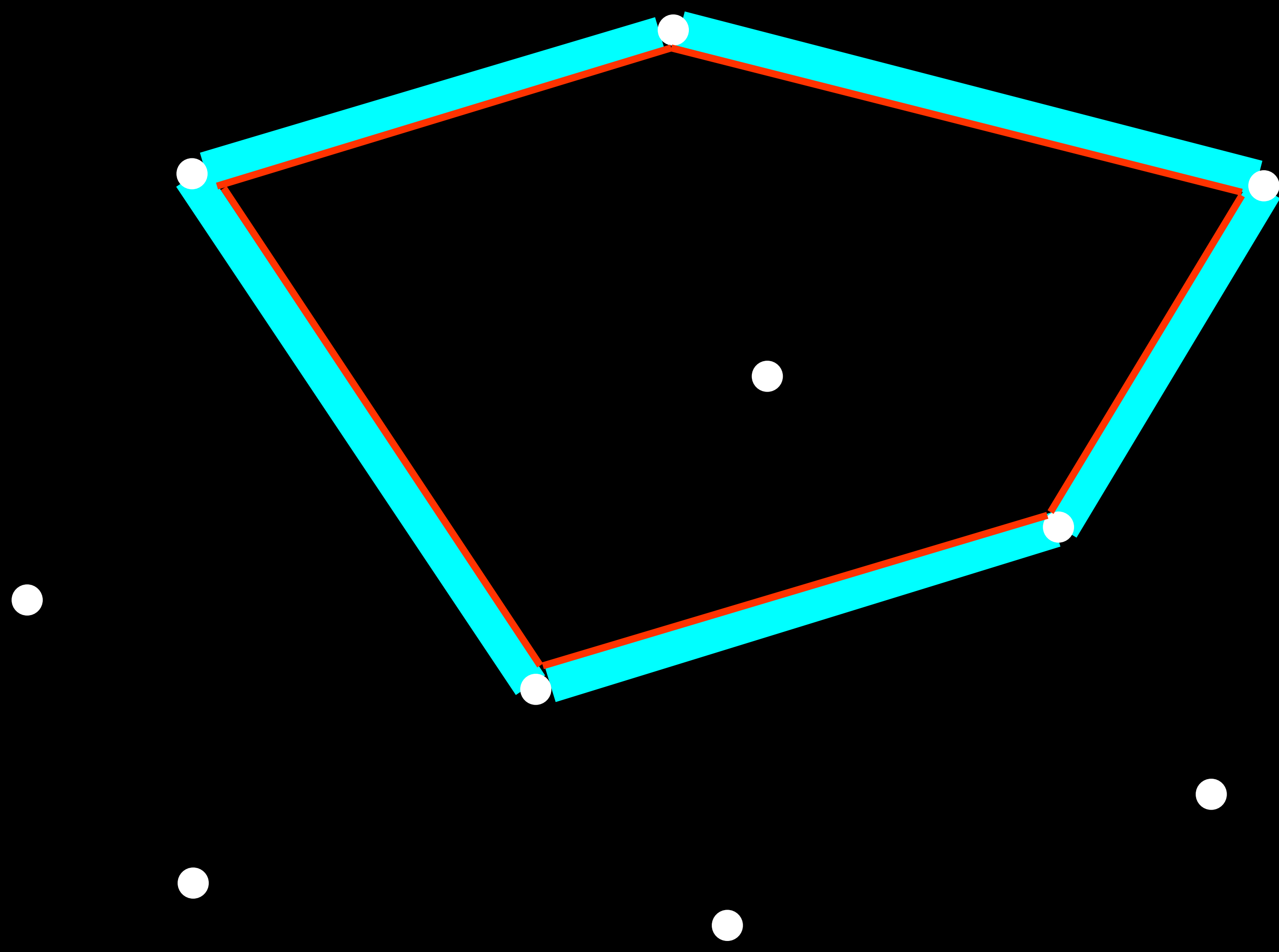
infinite graph sequence

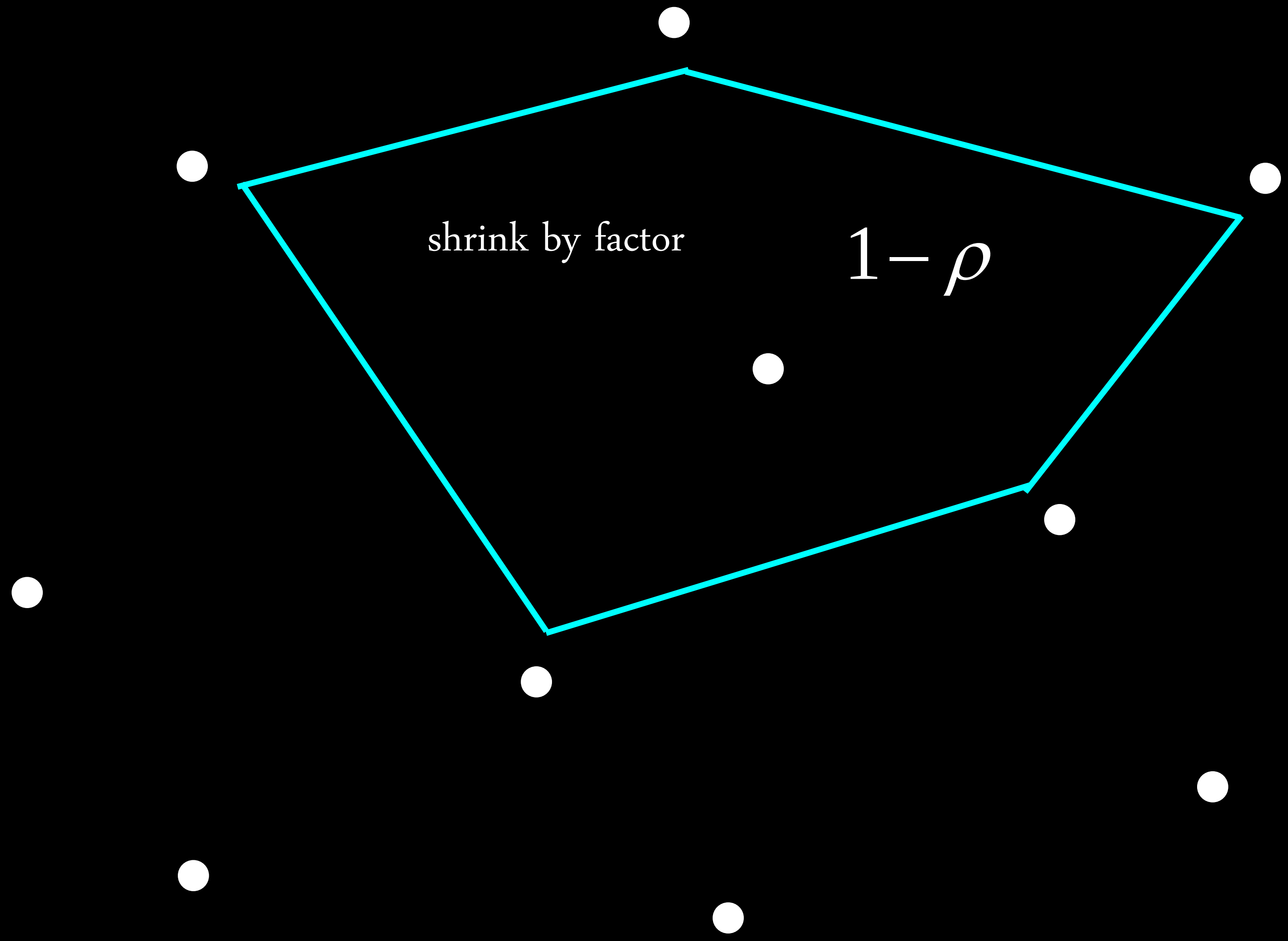


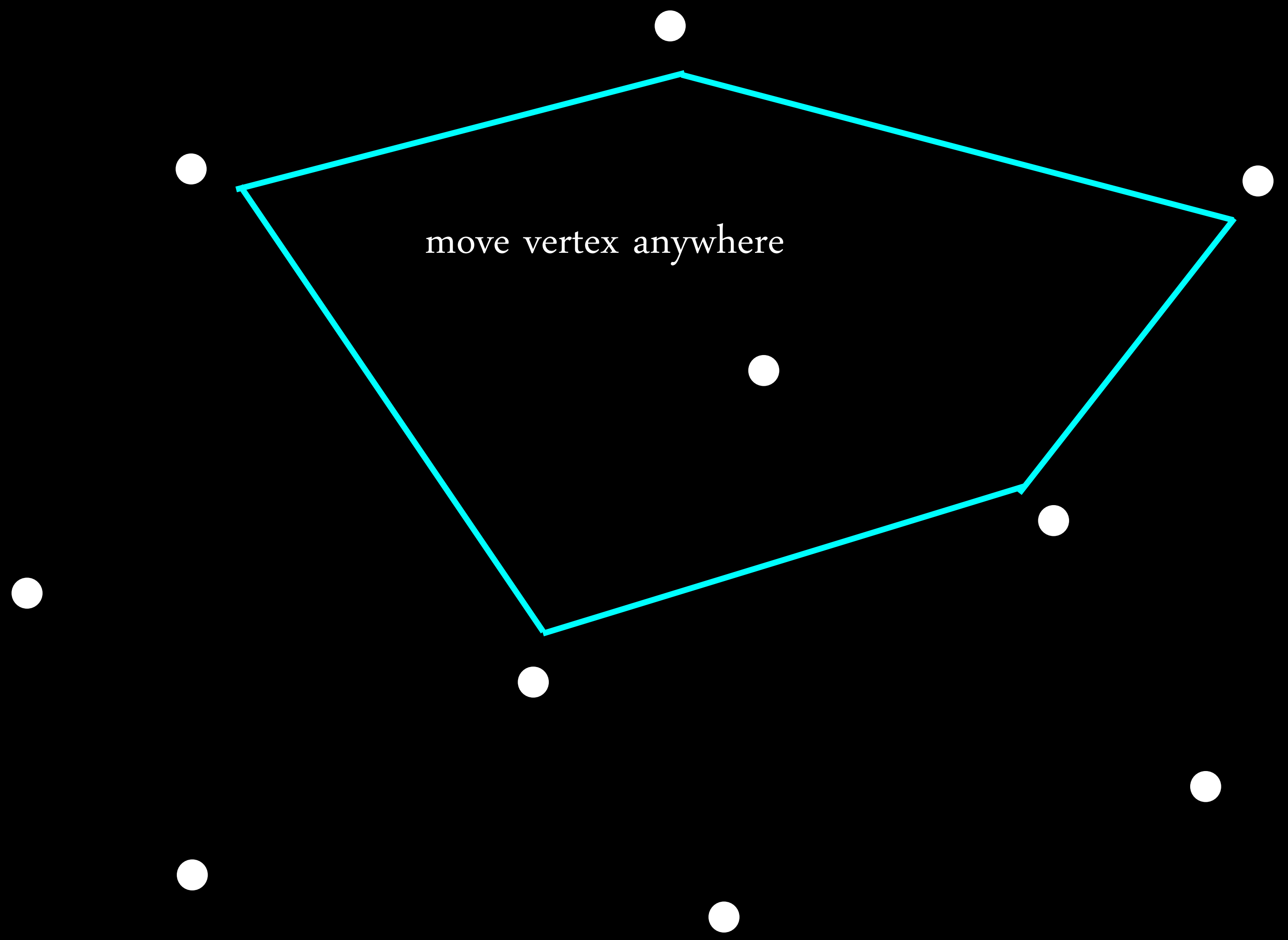


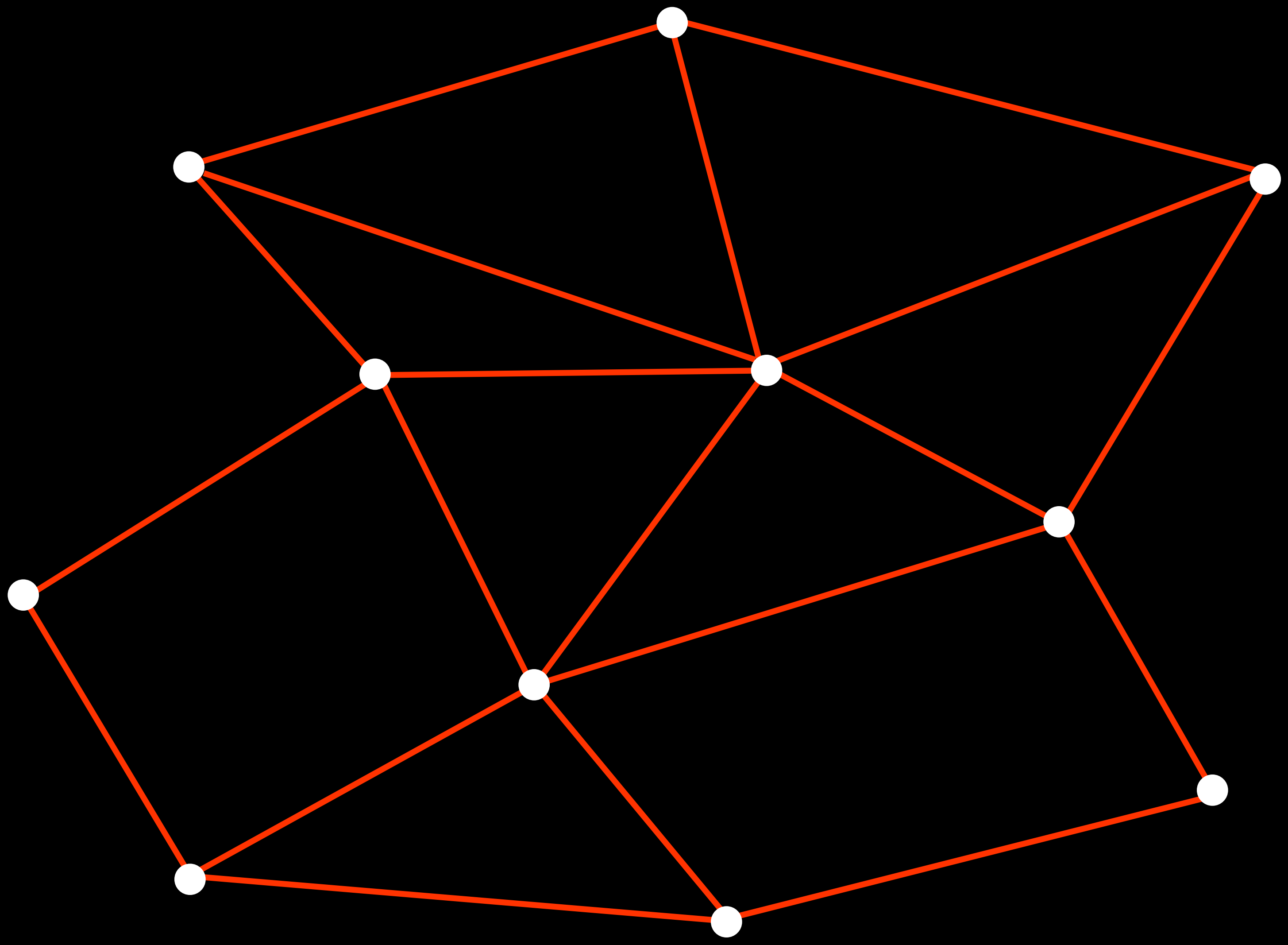




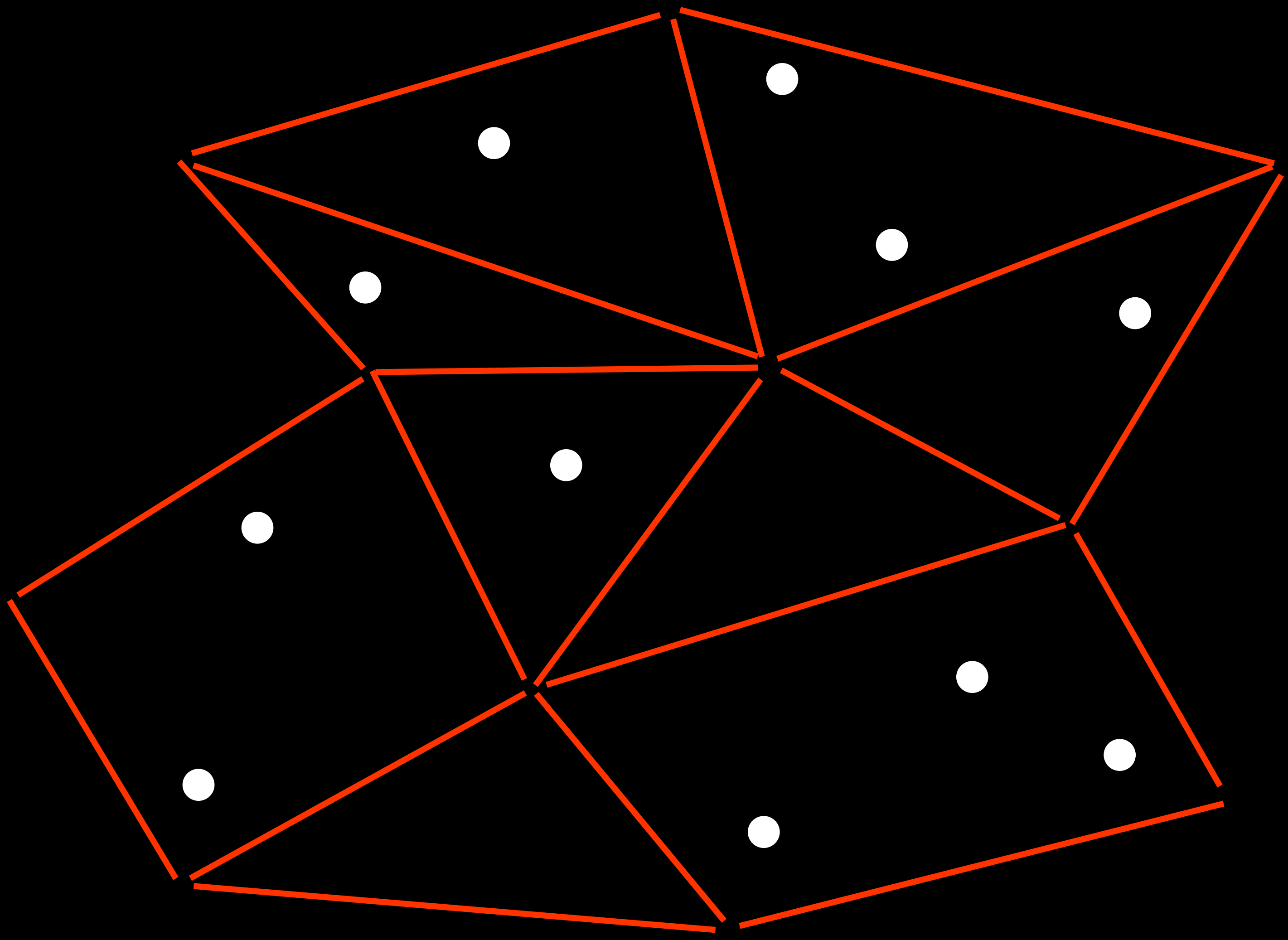


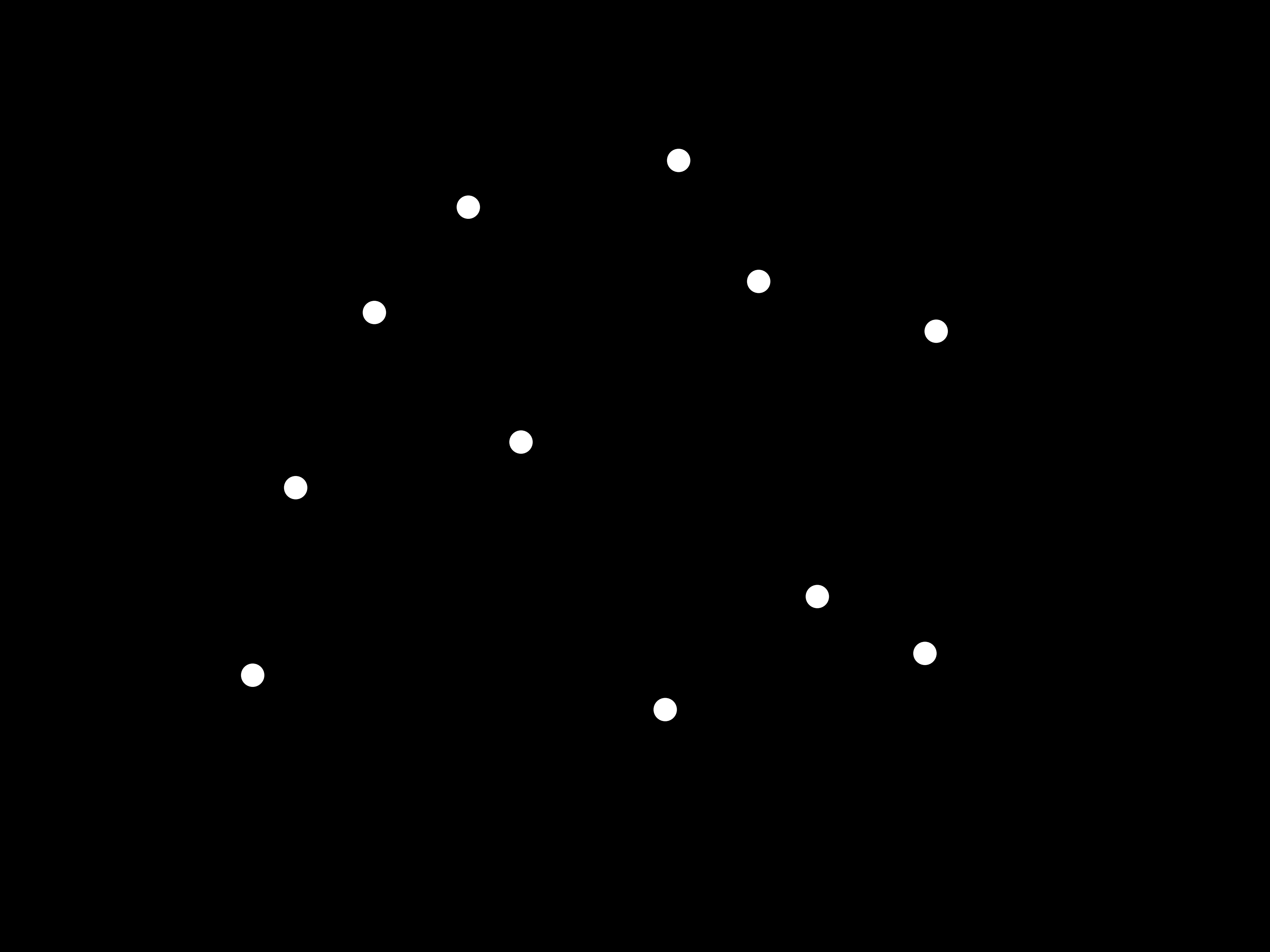

shrink by factor
$1-\rho$

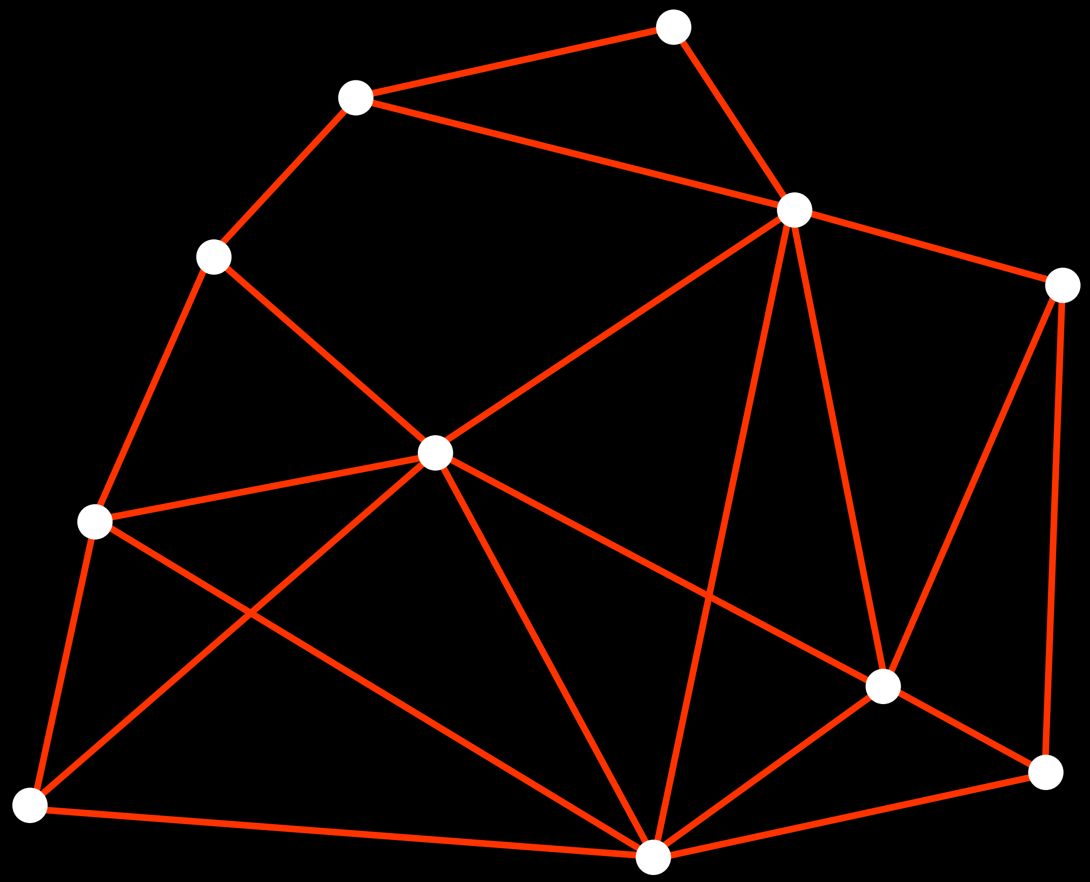

move vertex anywhere



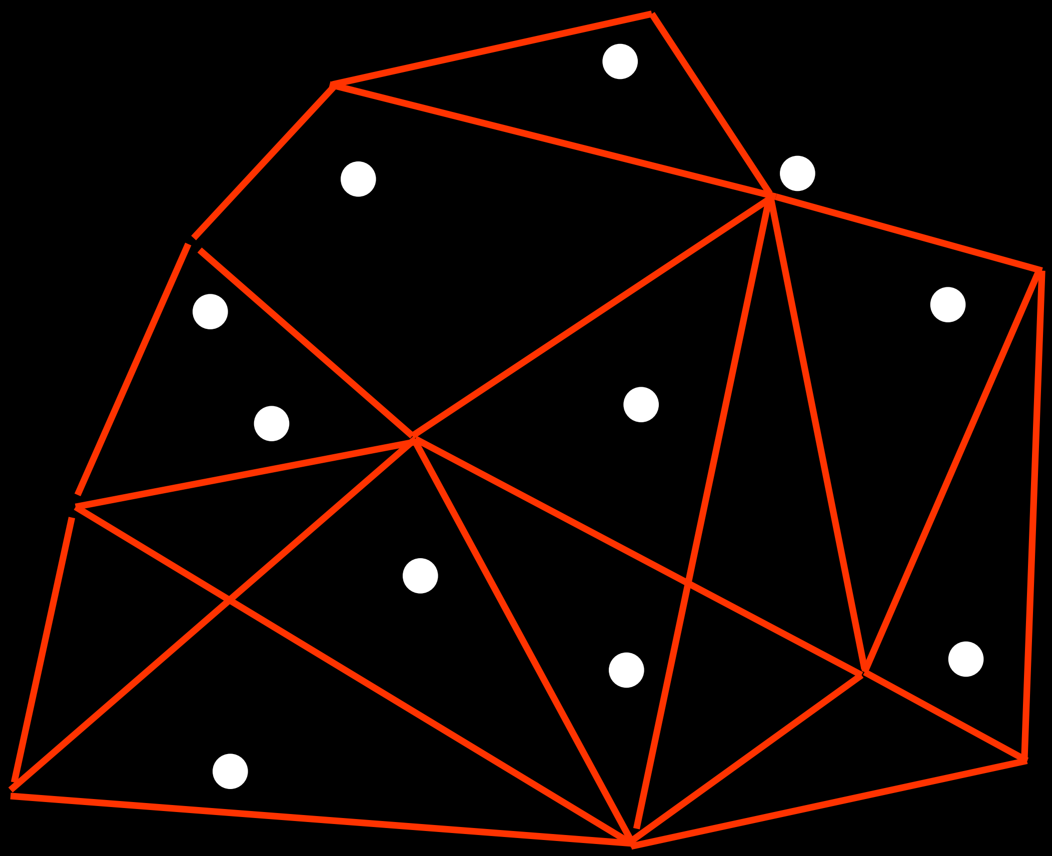
repeat for each node



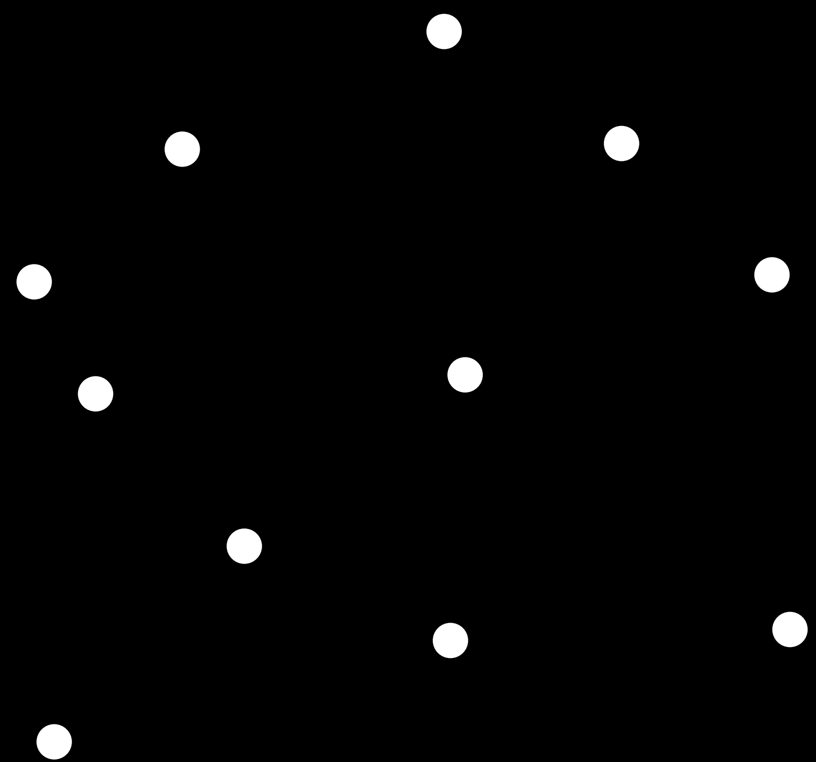




get second graph



move all the vertices



get 3rd graph, move vertices, repeat...

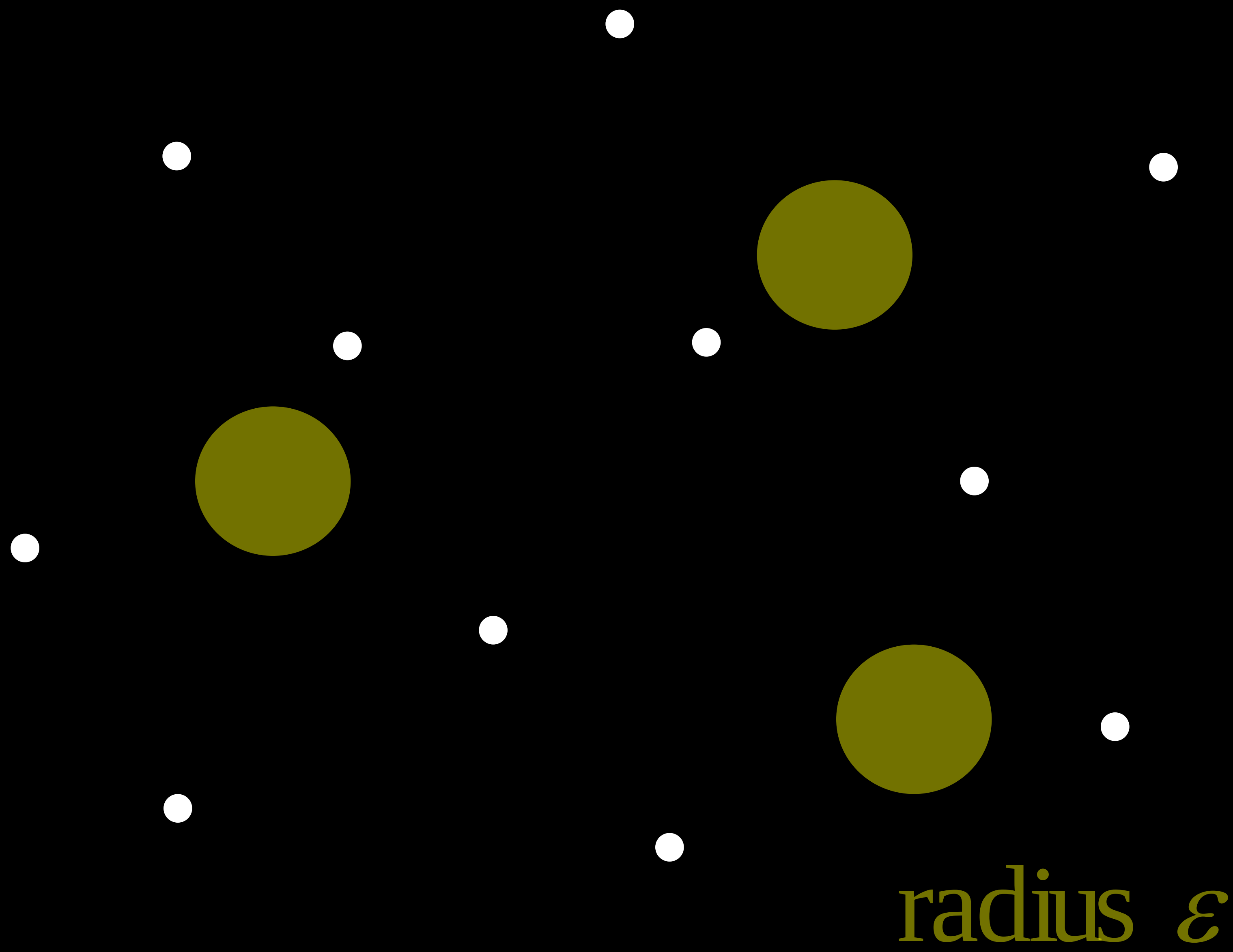
adversary chooses all graphs

and all moves nondeterministically

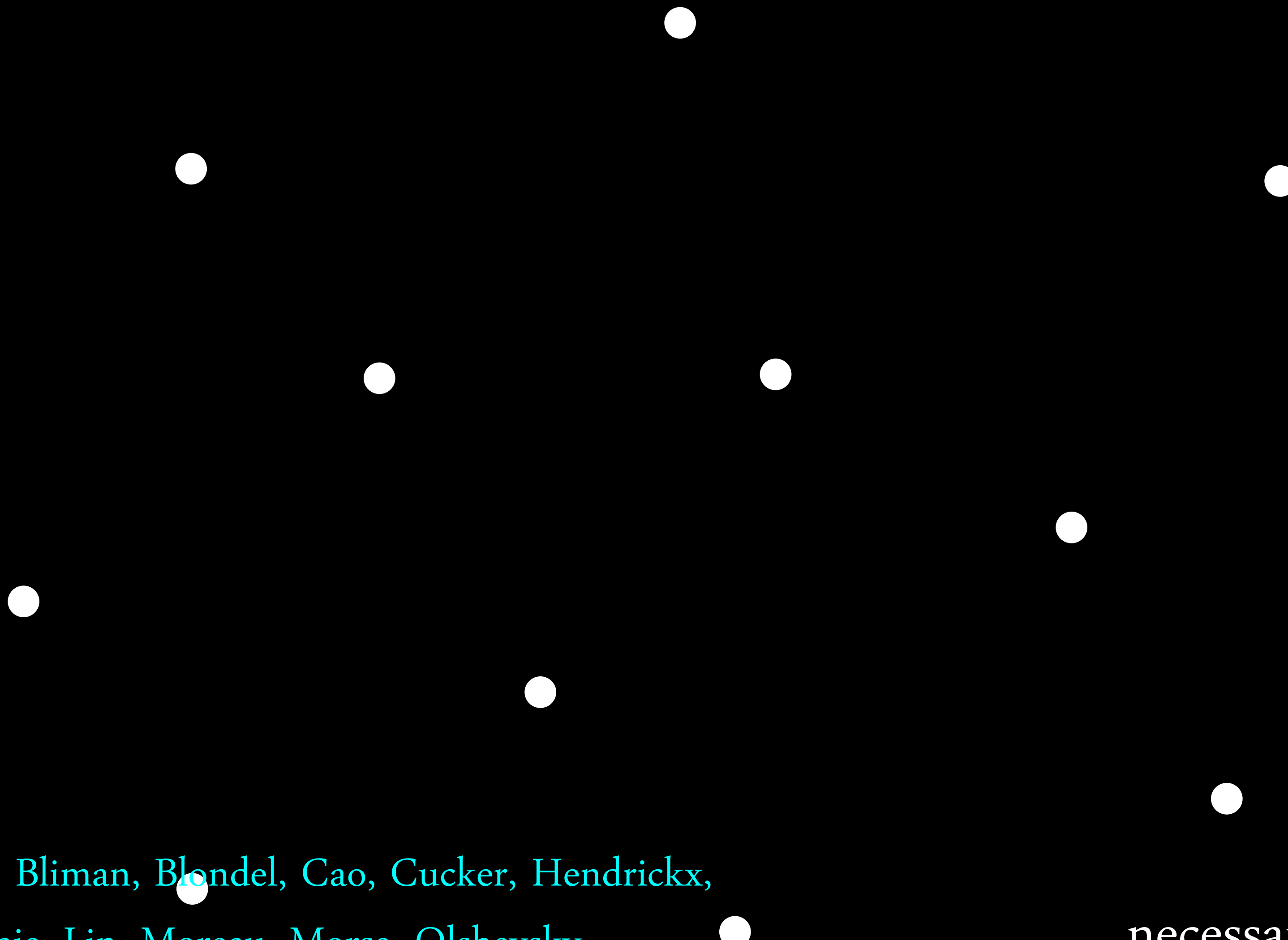
does this thing always converge?

yes

convergence



sync



Angeli, Bliman, Blondel, Cao, Cucker, Hendrickx,
Jadbabaie, Lin, Moreau, Morse, Olshevsky,
Spielman, Tsitsiklis, Wang, etc.

necessary conditions

no convergence time

THEOREM 1

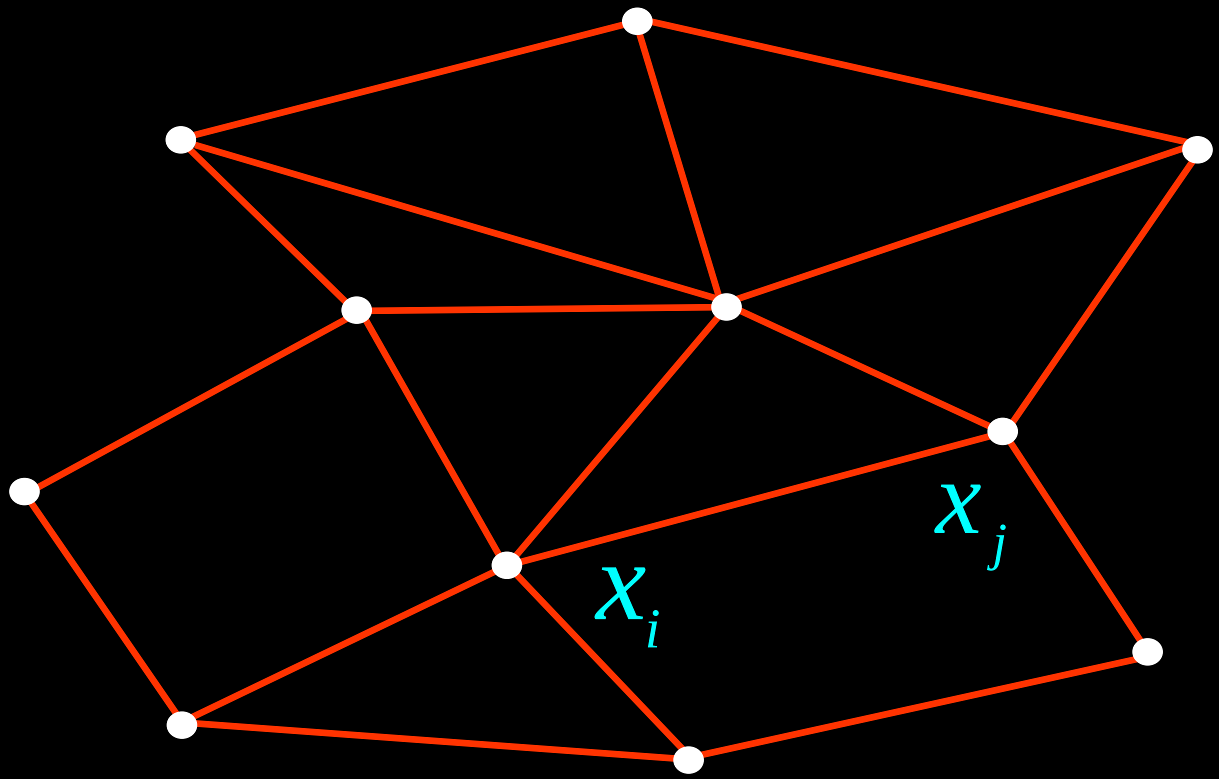
System always converges :

(i) number of moves is infinite

(ii) number of nontrivial moves

$$\leq \min \left\{ \varepsilon^{-1} \rho^{-O(n)} \;,\; (\log 1/\varepsilon)^{n-1} \rho^{-n^2 (1+o(1))} \right\}$$

main tool   total s-energy



$$E(s) = \sum_t \sum_{(i,j)\in G_t} \| x_i - x_j \|_2^s$$

$E(s)$

Dirichlet series → inverse formula

(Mellin transform) → lossless encoding

Does it ever converge?

$E(s)$

- converges for all $s > 0$
- analytic for $\Re s > 0$
- pole at $s = 0$ of order $n - 1$

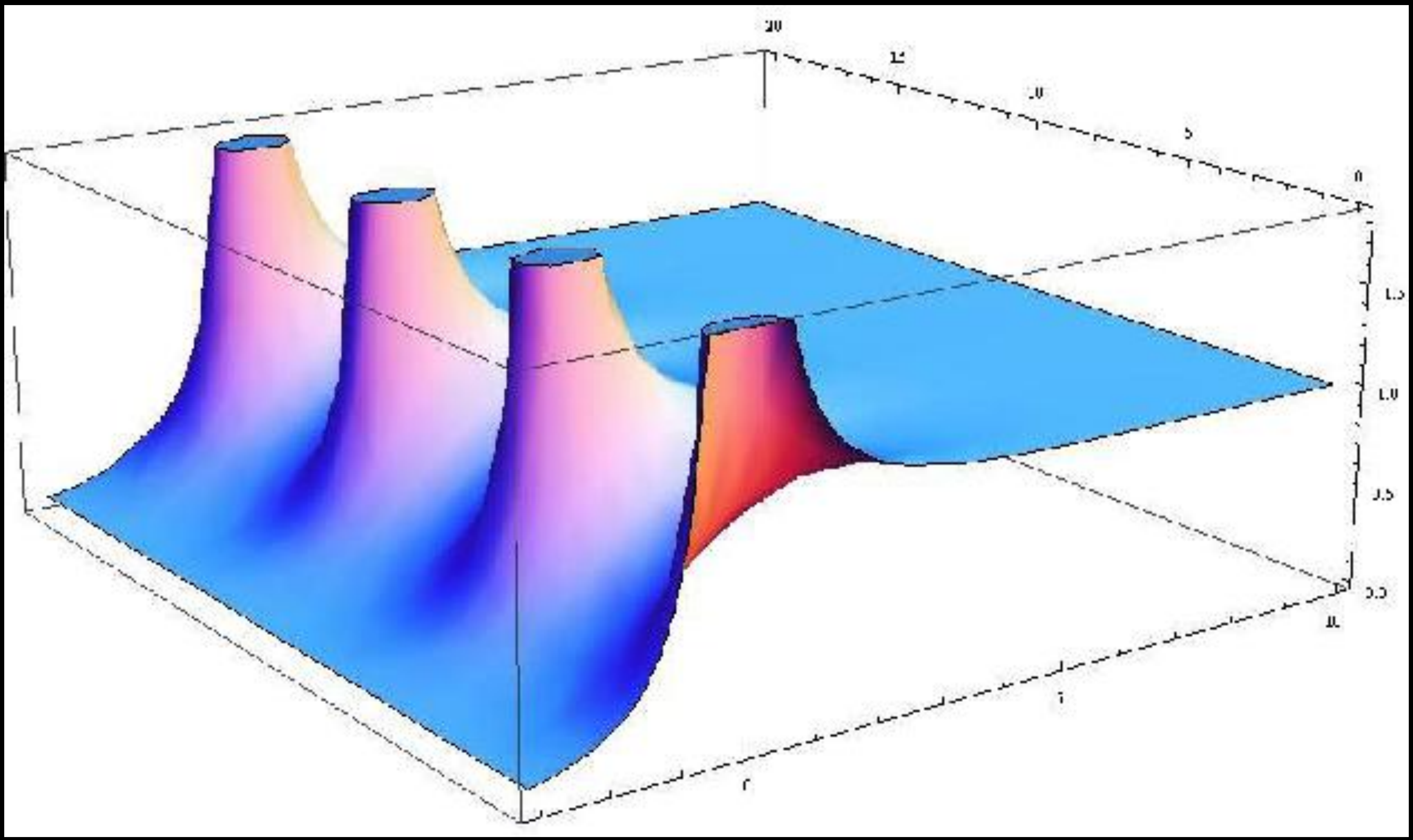
$E(s)$

- in general, no analytic continuation over whole plane

- conjecture

  for any n, max $E(s)$ has analytic continuation with discrete poles at s=0

true for $n = 2$



THEOREM 2

$$E(s) \leq \begin{cases} \rho^{-O(n)} & \text{if } s = 1 \\ s^{n-1} \rho^{-n^2(1+o(1))} & \text{if } s < 1 \end{cases}$$

proof algorithmicized proofs of old math

flow algorithm

recurrences, etc.

$$P = \underbrace{U^{-1} T U}$$

Schur's Lemma

Every square matrix is unitarily similar to a triangular matrix of $\mathbf{C}^n$. Apply the Gram–Schmidt orthonormalization procedure (0.6 to this basis to produce an orthonormal basis

$$x^{(1)}, z^{(2)}, \ldots, z^{(n)}$$

of $\mathbf{C}^n$. Array these orthonormal vectors left to right as the columns of unitary matrix U_1. Since the first column of AU_1 is $\lambda_1 x^{(1)}$, a calculatio reveals that $U_1^*(AU_1)$ has the form

$$U_1^*AU_1 = \left[\begin{array}{c|c} \lambda_1 & * \\ \hline 0 & A_1 \end{array}\right]$$

The matrix $A_1 \in M_{n-1}$ has eigenvalues $\lambda_2, \ldots, \lambda_n$. Let $x^{(2)} \in \mathbf{C}^{n-1}$ be normalized eigenvector of A_1 corresponding to λ_2, and do it all ov again. Determine a unitary $U_2 \in M_{n-1}$ such that

$$U_2^*A_1U_2 = \left[\begin{array}{c|c} \lambda_2 & * \\ \hline 0 & A_2 \end{array}\right]$$

and let

$$V_2 = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & U_2 \end{array}\right]$$

The matrices V_2 and U_1V_2 are then unitary, and $V_2^*U_1^*AU_1V_2$ has the form

$$V_2^*U_1^*AU_1V_2 = \left[\begin{array}{cc|c} \lambda_1 & * & * \\ 0 & \lambda_2 & \\ \hline & 0 & A_2 \end{array}\right]$$

Continue this reduction to produce unitary matrices $U_i \in M_{n-i+1}$, $i = \ldots, n-1$ and unitary matrices $V_i \in M_n$, $i = 2, \ldots, n-1$. The matrix

$$U = U_1V_2V_3 \cdots V_{n-1}$$

unitary and U^*AU yields the desired form.

If all eigenvalues of $A \in M_n(\mathbf{R})$ happen to be real, then the corre onding eigenvectors can be chosen to be real and all the above steps y be carried out in real arithmetic, verifying the final assertion. □

Remark: Follow the proof of (2.3.1) to see that "upper triangular" ld be replaced by "lower triangular" in the statement of the theorem h, of course, a different unitary equivalence U.

2 **Example.** Neither the unitary matrix U nor the triangular matrix Theorem (2.3.1) is unique. Not only may the diagonal entries of T

the eigenvalues of A) appear in any order, but unitarily equivalent upp triangular matrices may appear very different above the diagonal. F example,

$$T_1 = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} 2 & -1 & 3\sqrt{2} \\ 0 & 1 & \sqrt{2} \\ 0 & 0 & 3 \end{bmatrix}$$

are unitarily equivalent via

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

In general, many different upper triangular matrices can be in the sam unitary equivalence class.

Remark: Notice that the technique of the proof (2.3.1) is simply th of sequential deflation, as outlined in Problem 8 in Section (1.4).

Exercise. If $A \in M_n$ is unitarily equivalent to an upper triangular matr $T = [t_{ij}] \in M_n$, the entries t_{ij} are not uniquely determined, but the quan ity $\sum_{i<j} |t_{ij}|^2$ is uniquely determined. Determine the value of $\sum_{i<j} |t_{ij}$ in terms of the entries and eigenvalues of A. *Hint:* Use (2.2.2).

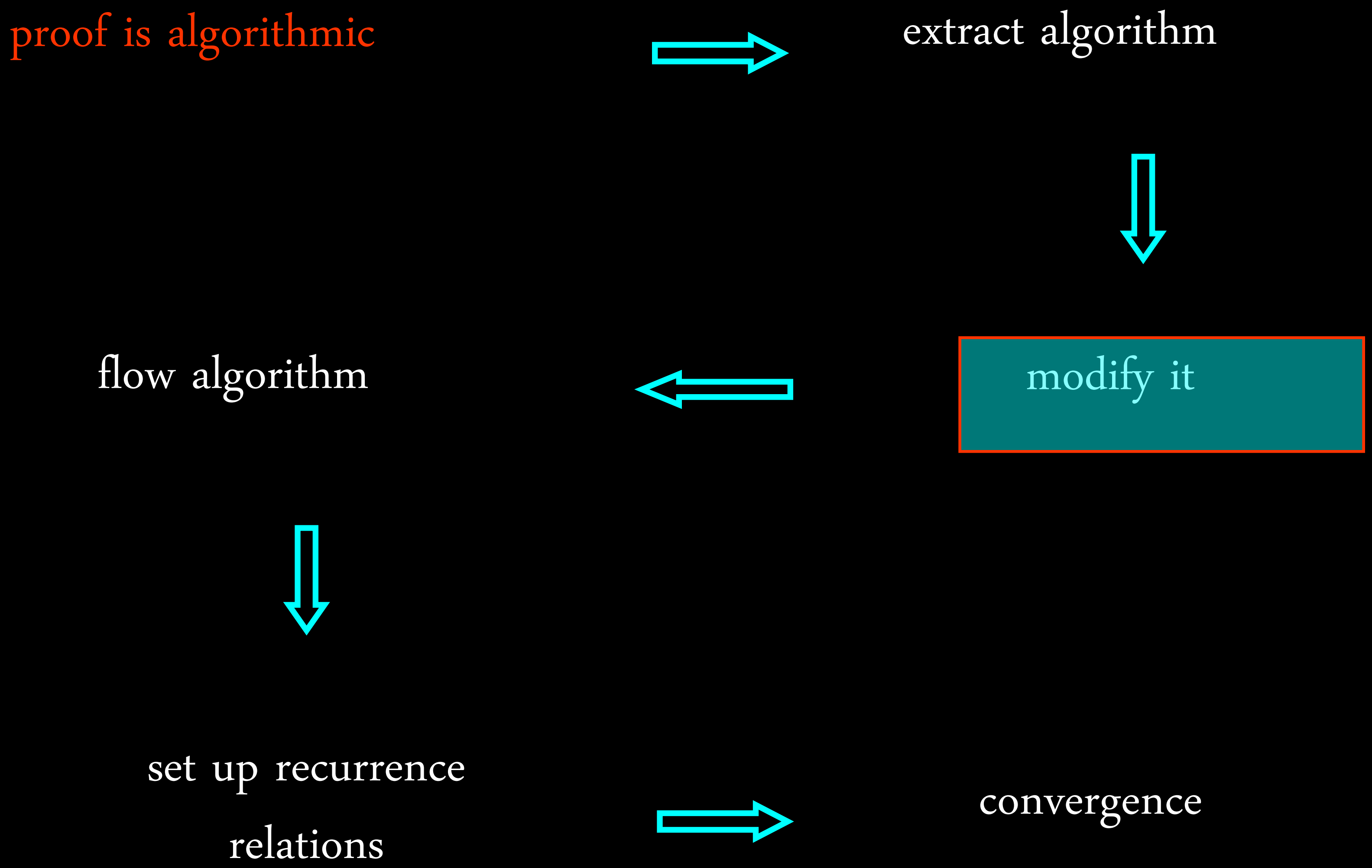
Exercise. If $A = [a_{ij}]$ and $B = [b_{ij}] \in M_2$ are similar and if $\sum_{i,j} |a_{ij}|^2$ $\sum_{i,j} |b_{ij}|^2$, show that A and B are unitarily equivalent. Show by examp that this is not the case in higher dimensions. *Hint:* Notice that if A and are unitarily equivalent, then so are $A + A^*$ and $B + B^*$. Consider

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

It is a useful adjunct to (2.3.1) that a commuting family of matrice may be simultaneously upper triangularized.

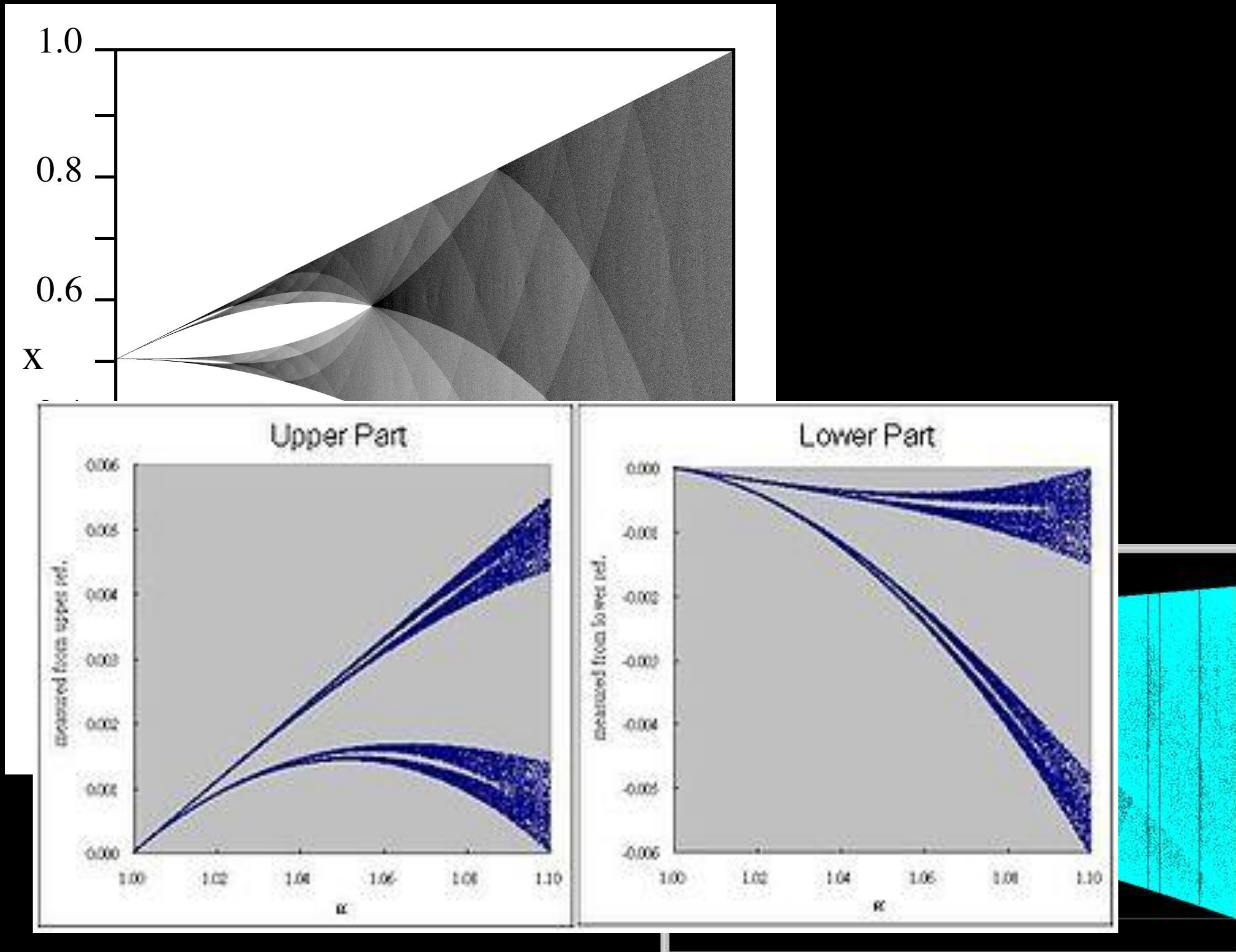
2.3.3 **Theorem.** Let $\mathcal{F} \subseteq M_n$ be a commuting family. There is a unitar matrix $U \in M_n$ such that U^*AU is upper triangular for every $A \in \mathcal{F}$.

Proof: Return to the proof of (2.3.1). Exploiting (1.3.17) at each ste of the proof in which a choice of an eigenvector (and unitary matrix is made, the *same* eigenvector (and unitary matrix) may be chosen fo every $A \in \mathcal{F}$. Moreover, unitary equivalence preserves commutativity proof is algorithmic


proof is algorithmic
extract algorithm
modify it
flow algorithm
set up recurrence relations
convergence

use algorithms to analyze algorithms

…and stay warm


1.0
0.8
0.6
x
Upper Part
0.006
0.005
0.004
0.003
0.002
0.001
0.000
1.00
1.02
1.04
1.06
1.08
1.10
Lower Part
0.000
-0.001
-0.002
-0.003
-0.004
-0.005
-0.006
1.00
1.02
1.04
1.06
1.08
1.10