Market Equilibrium under Separable, Piecewise-Linear, Concave Utilities

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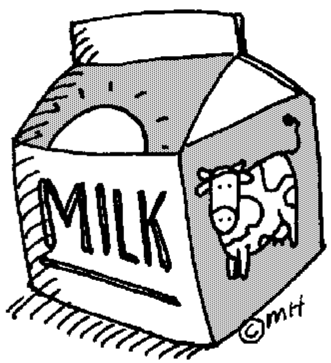
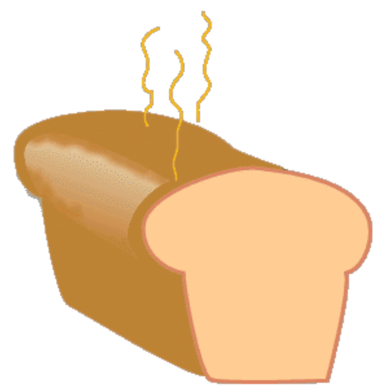
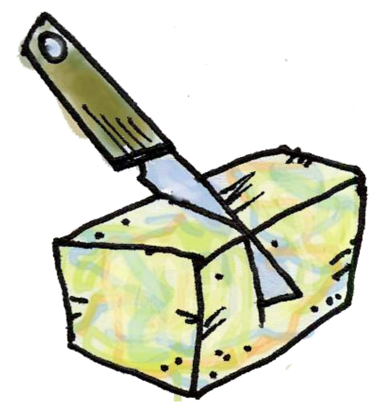
Mihalis Yannakakis

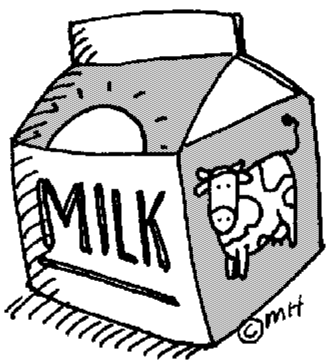
Fisher's model with plc utilities








TM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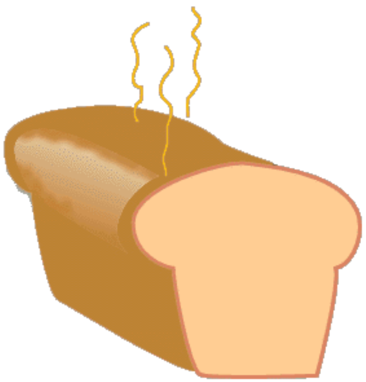
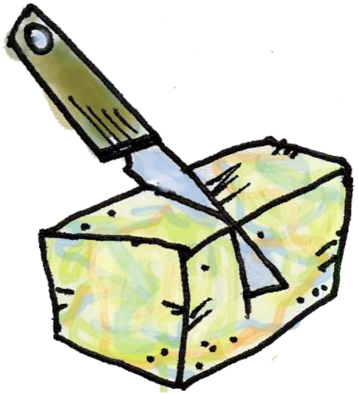
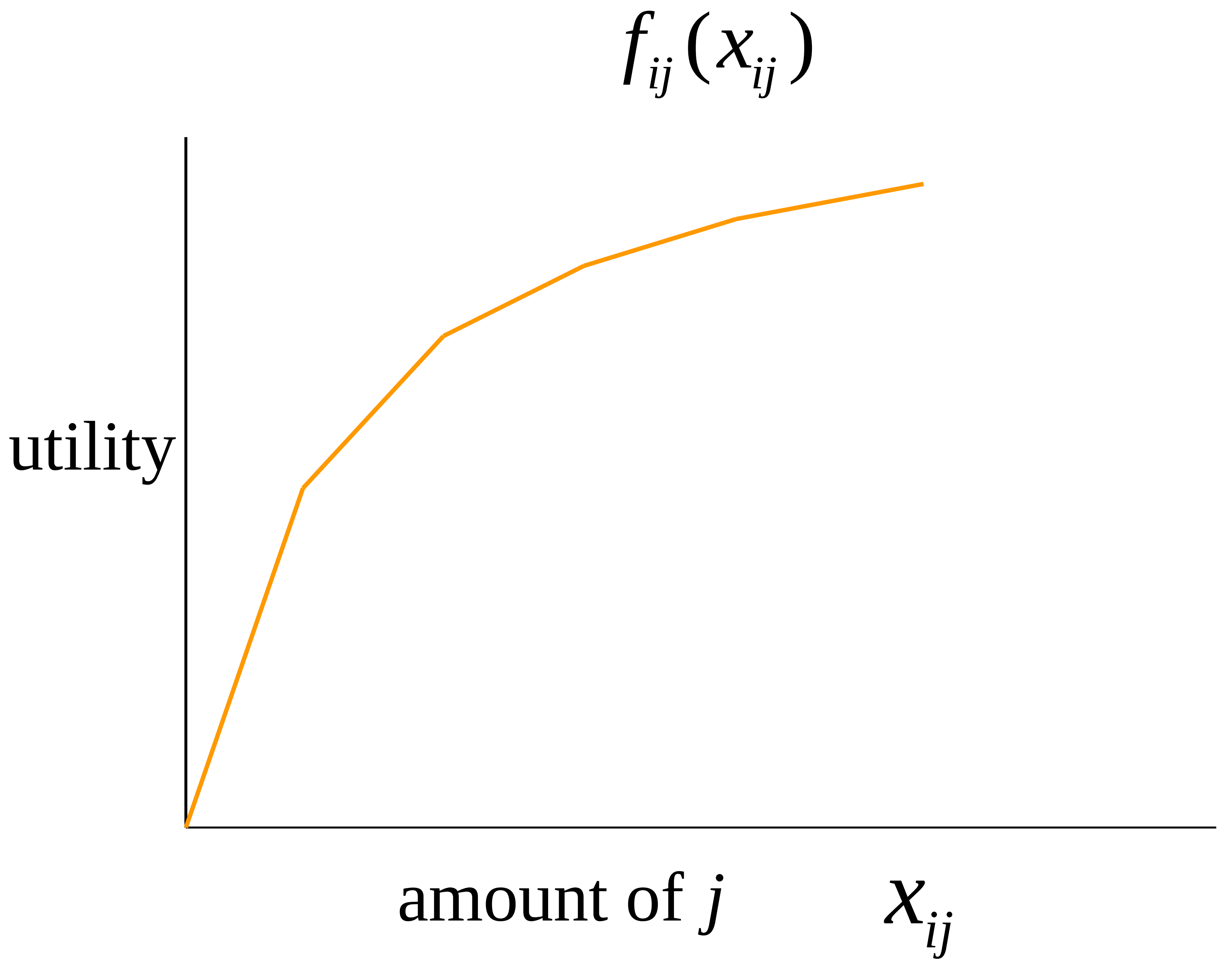


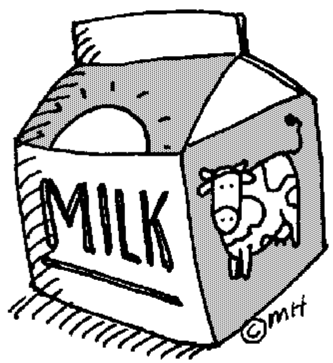


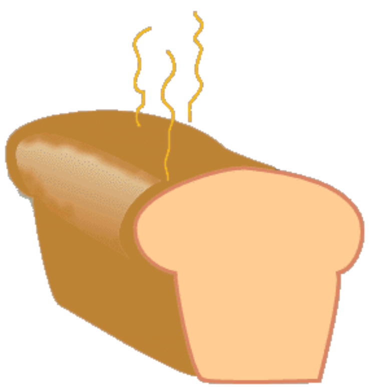


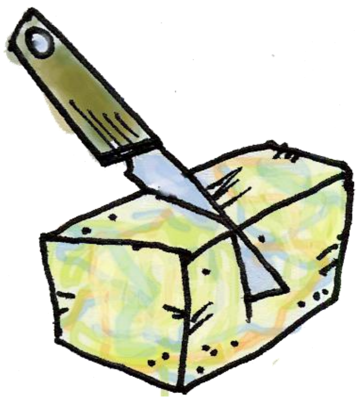
$f_{ij}(x_{ij})$
utility
amount of j
x_{ij}

total utility

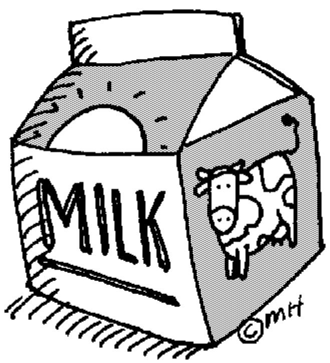
$$v_i = \sum_{j \in G} f_{ij}(x_{ij})$$



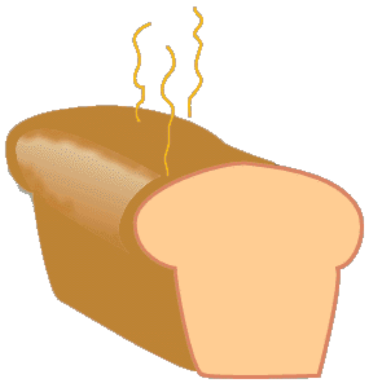




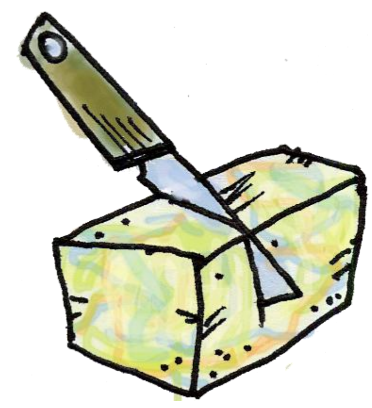
For given prices,
find optimal bundle of goods



p_1



p_2

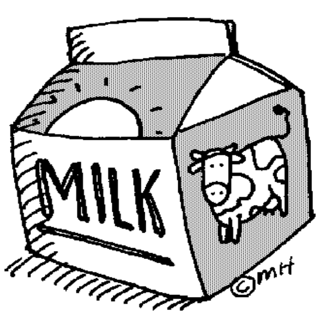


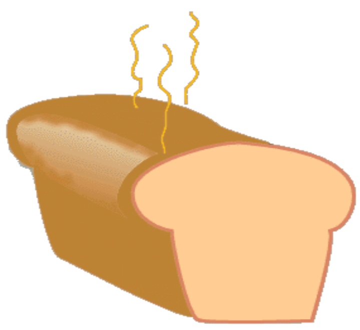
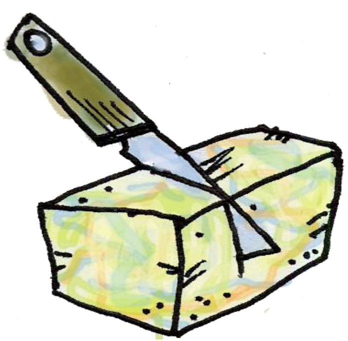
p_3

Several buyers with different utility functions and moneys.





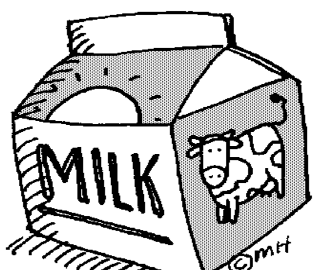


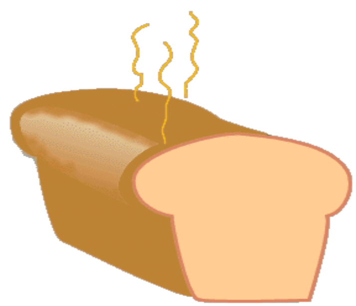



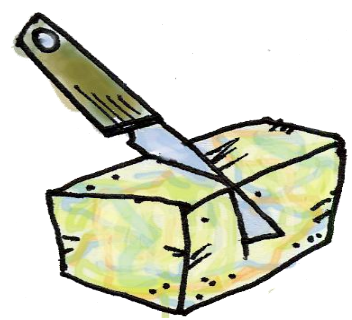
Several buyers with different utility functions and moneys.

Find equilibrium prices.







p_1 p_2 p_3

Long-standing open problem

- Complexity of finding an equilibrium for Fisher and Arrow-Debreu models under separable, plc utilities?

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- 2009: Both PPAD-complete

Linear Fisher market

- DPSV, 2002:

  Combinatorial, polynomial time algorithm

- Assume:
  - m_i : money of buyer *i*.
  - One unit of each good *j*.

Eisenberg-Gale Program, 1959

$$\max \sum_i m_i \log v_i$$

$$s.t.$$

$$\forall i: \quad v_i = \sum_j u_{ij} x_{ij}$$

$$\forall j: \quad \sum_i x_{ij} \leq 1$$

$$\forall ij: \quad x_{ij} \geq 0$$

Eisenberg-Gale Program, 1959

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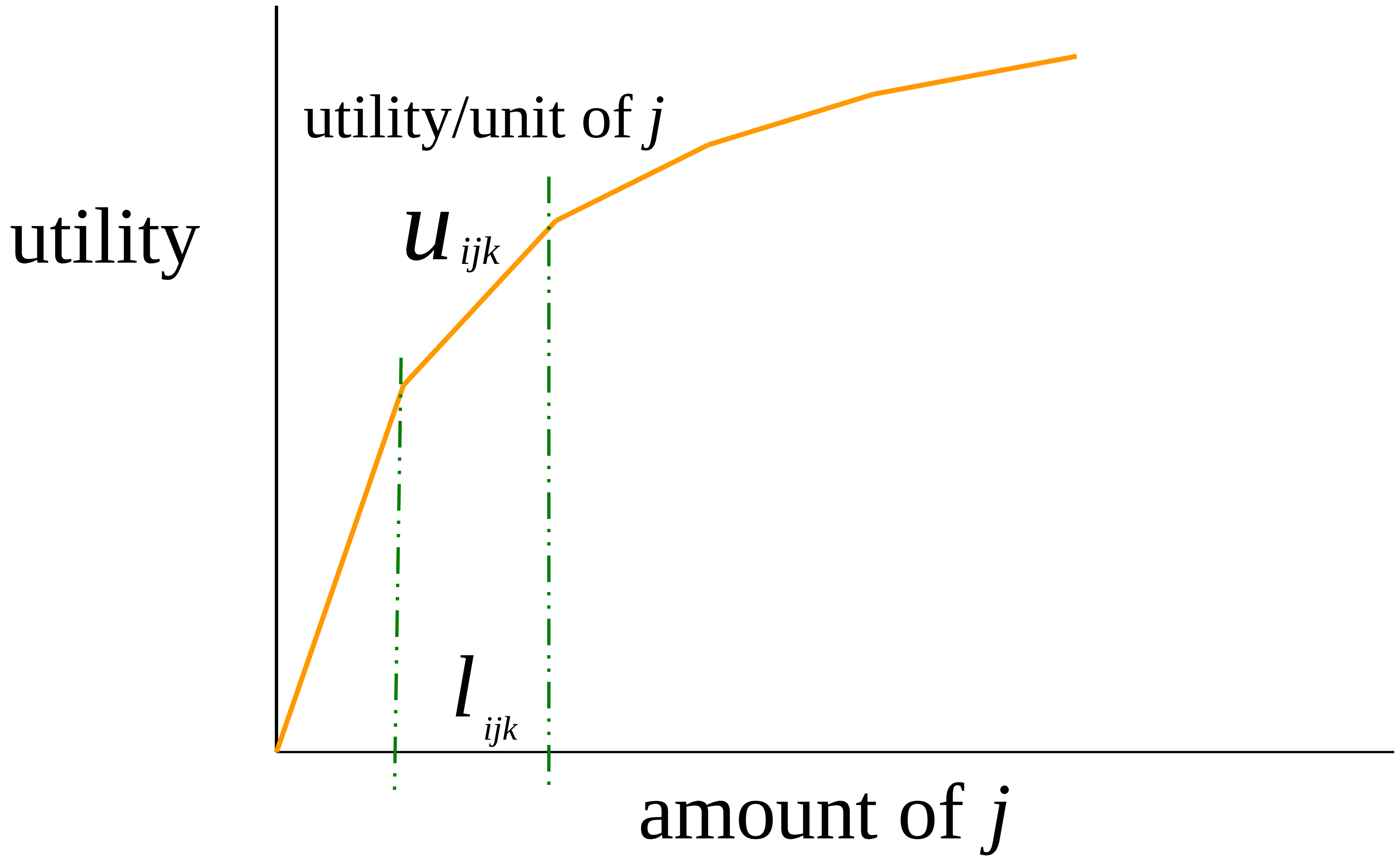
prices p_j

$$\forall ij: \quad x_{ij} \ge 0$$

Generalize EG program to piecewise-linear, concave utilities?



Generalization of EG program

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Markets with piecewise-linear, concave utilities

- V. & Yannakakis, 2007

Equilibrium is rational for both
Fisher and Arrow-Debreu models

A simpler question

- Given prices $\underline{p}$, are they equilibrium prices?

- What does buyer i's optimal bundle look like?

Bang-per-buck w.r.t. $\underline{p}$

- bpb(s) $= \dfrac{u_{ijk}}{p_j}$
- For each buyer *i*:
  - Sort segments by decreasing bpb, and partition by equality.
- Allocate in order: $Q_1, \; Q_2, \; Q_3, \; \ldots$

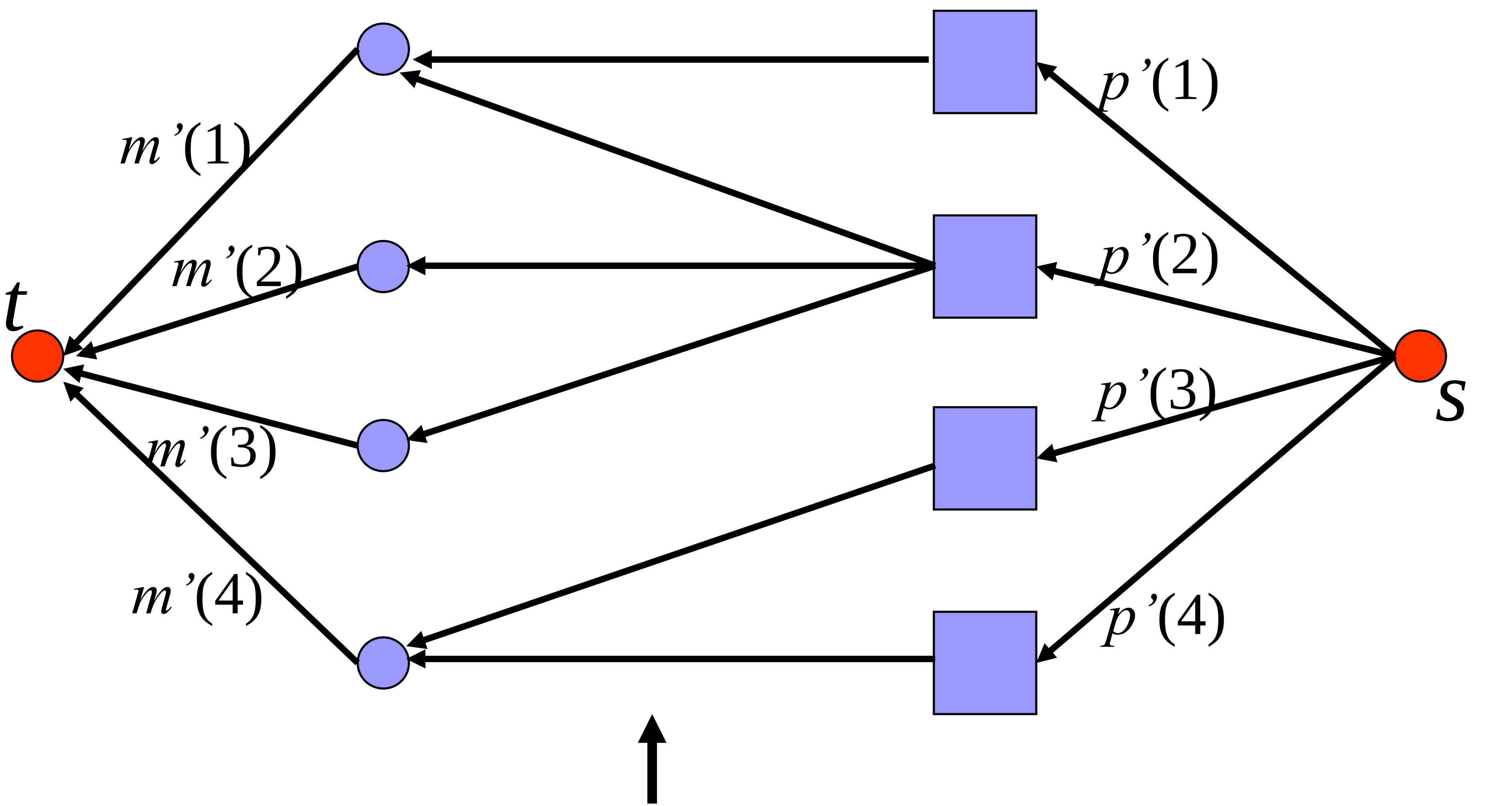
- Find min k s.t. $Q_1, Q_2, \ldots Q_k$ exhaust buyer i' s money

- $Q_1, Q_2, \ldots Q_{k-1} \quad Q_k \quad Q_{k+1,} \cdots$

  forced flexible undesirable

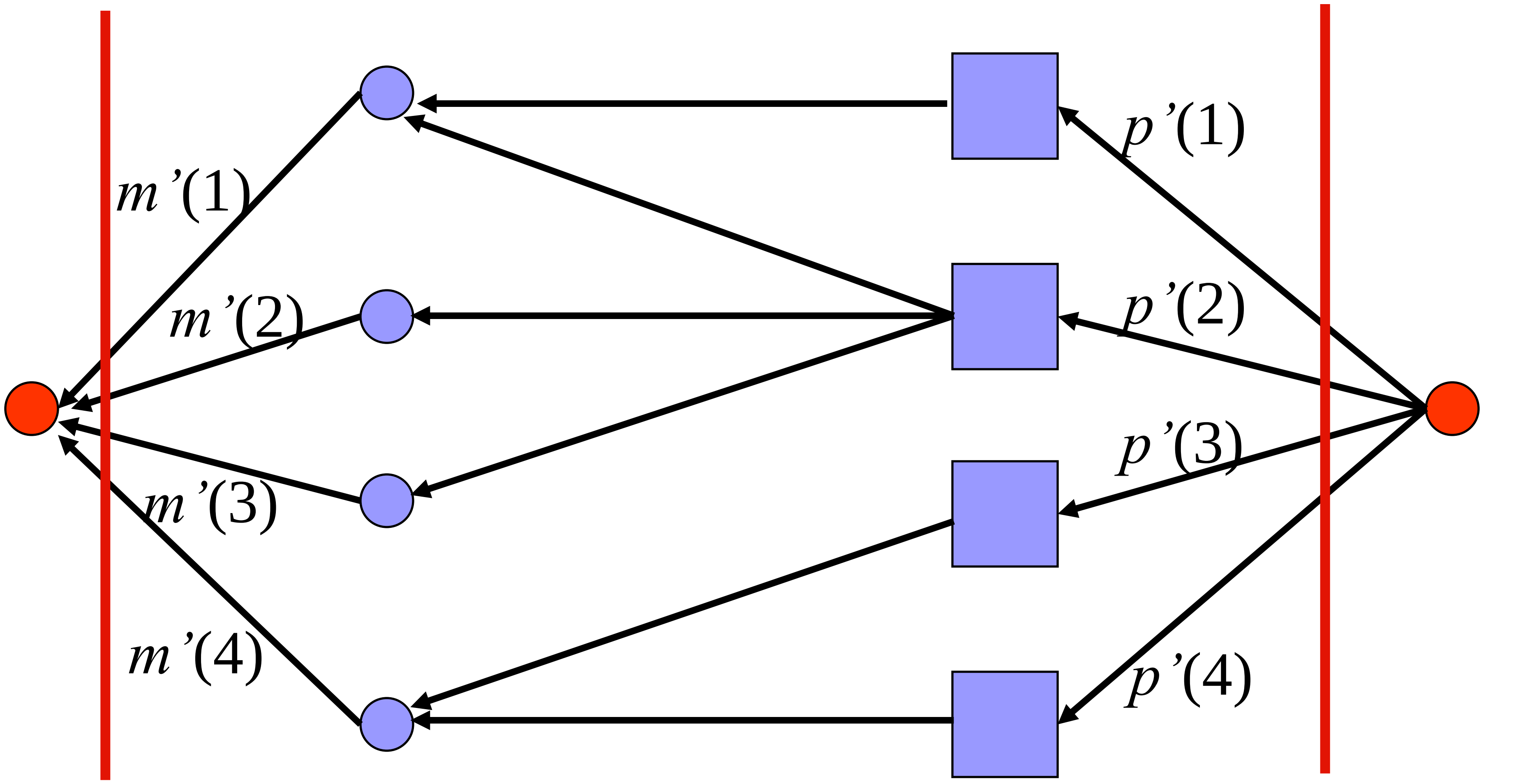
- After forced allocations:
  - *m'(i)*: *i*'s left-over money
  - *p'(j)*: left-over value of good *j*.

Network $N(\underline{p})$



flexible partitions

Max flow in *N(p)*



p: equilibrium prices iff both cuts saturated

- Can write max-flow in *N* as an LP
  - Introduce flow variables

- Suppose, corresponding to an equilibrium, we "guess" flexible partitions. Don't know *p*

- Pick prices as variables. Construct *N*.
  - All edge capacities linear in price variables.

- Write max-flow LP (it is still linear!)
  - Introduce flow variables.

Rationality proof

- If "guess" is correct,
  optimal solution to LP yields equilibrium prices.

- Hence rational!

Rationality proof

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  optimal solution to LP yields equilibrium prices.
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- In P??

NP-hardness does not apply

- Megiddo, 1988:
  - Equilibrium NP-hard => NP = co-NP
- Papadimitriou, 1991: PPAD
  - 2-player Nash equilibrium is PPAD-complete
  - Rational
- Etessami & Yannakakis, 2007: FIXP
  - 3-player Nash equilibrium is FIXP-complete
  - Irrational

Markets with piecewise-linear, concave utilities

- Chen, Dai, Du, Teng, 2009:
  - PPAD-hardness for Arrow-Debreu model

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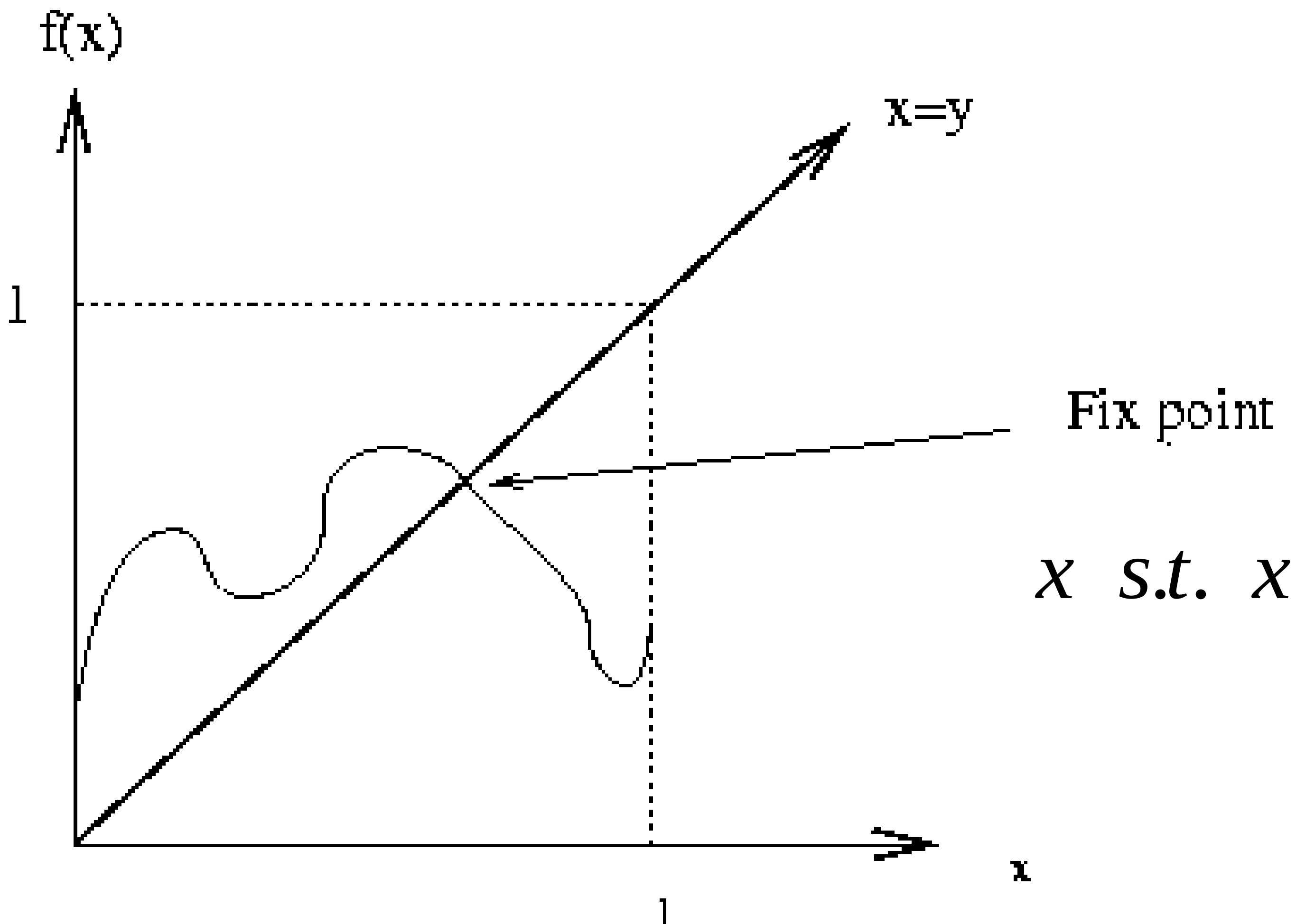
- Chen, Teng, 2009:
  - PPAD-hardness for Fisher's model
- V., & Yannakakis, 2009:
  - PPAD-hardness for Fisher's model
  - Membership in PPAD for both models,

Sufficient condition: non-satiation.

Markets with piecewise-linear, concave utilities

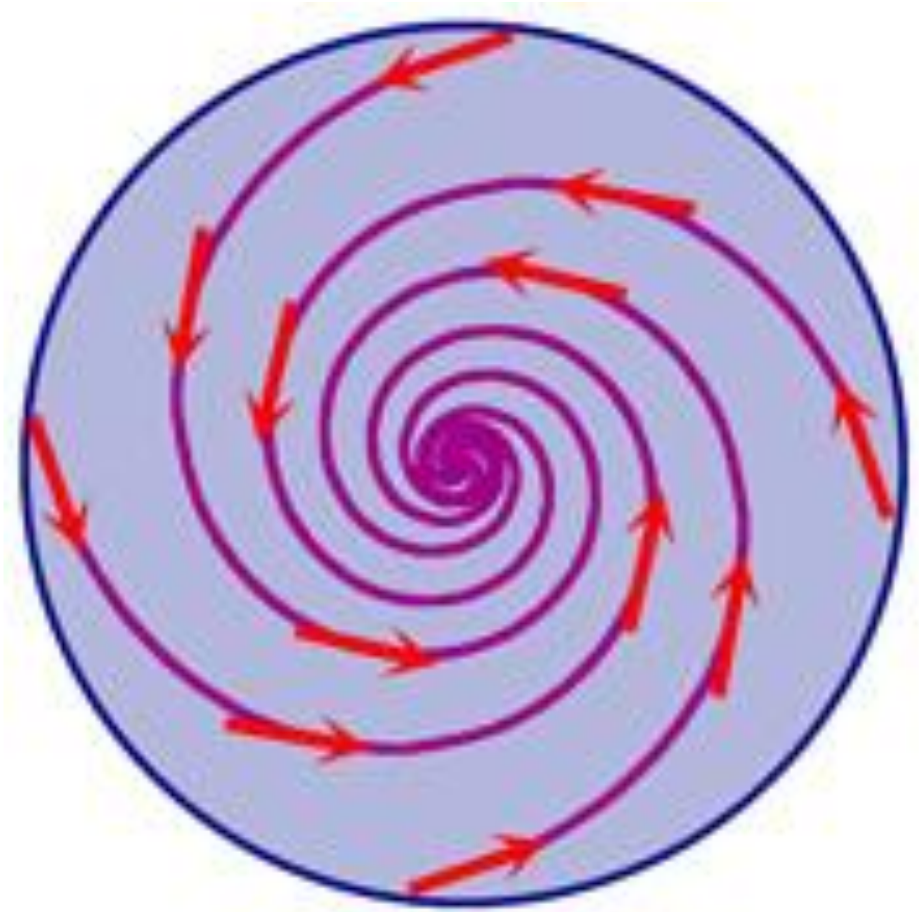
- Chen, Dai, Du, Teng, 2009:
  - PPAD-hardness for Walras' model
- Chen, Teng, 2009:
  - PPAD-hardness for Fisher's model
- V., & Yannakakis, 2009:
  - PPAD-hardness for Fisher's model
  - Membership in PPAD for both models,
  - Otherwise, deciding existence of eq. is NP-hard

Brouwer's Fixed Point Theorem



$$x \ \ s.t. \ \ x = f\left(x\right)$$

Brouwer's Fixed Point Theorem



Kakutani's fixed point theorem

- S: compact, convex set in R^n

$$f : S \to 2^S$$

upper hemi-continuous

$$\exists x \ s.t. \ x \in f(x)$$

- FIXP = fixed points of Brouwer functions represented as polynomially computable algebraic circuits over basis {+,-,*,/,max,min} with rationals.

- EY, 2007:
  PPAD = fixed point of polynomial time piecewise-linear Brouwer function.

- *M*: instance satisfying sufficient conditions.
- Prove: equilibrium for *M* can be found in PPAD
- Geanakoplos, 2003: Brouwer function-based proof of existence of equilibrium

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- Prove: equilibrium for *M* can be found in PPAD
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- Piecewise-linear Brouwer function??

Proof of membership in PPAD

- *F:* correspondence used in Kakutani-based proof of existence of equilibrium.
- *G:* piecewise-linear Brouwer approximation of *F*
- *(p*, x*):* fixed point of *G*.

Can be found in PPAD

Yields "guess"

Proof of membership in PPAD

- *F*: correspondence used in Kakutani-based proof of existence of equilibrium.
- *G*: piecewise-linear Brouwer approximation of *F*
- *(p*, x*)*: fixed point of *G*.
  Can be found in PPAD and yields "guess"
- Solve rationality LP to find equilibrium for *M*!

Algorithmic ratification of the "invisible hand of the market"??

How do we salvage the situation?

Is PPAD really hard?

What is the "right" model?

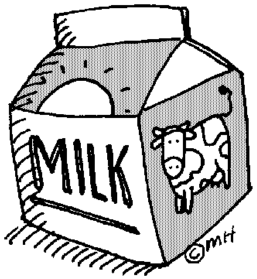
Price discrimination markets

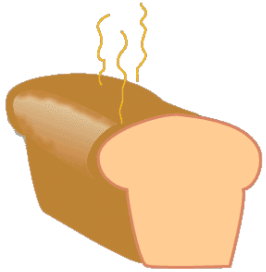
- Goel & V., 2009:

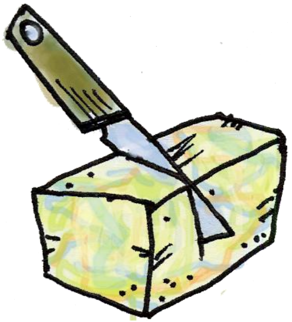
  Perfect price discrimination market.

  Business charges each consumer what they are willing and able to pay.

plc utilities







- Middleman buys all goods and sells to buyers, charging according to utility accrued.
  - Given $\underline{p}$, there is a well defined rate for each buyer.

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- Equilibrium is captured by a convex program –
  - Efficient algorithm for equilibrium

Generalization of EG program works!

$$\max \sum_i m_i \log v_i$$

$$s.t.$$

$$\forall i: \quad v_i = \sum_{j,k} u_{ijk} x_{ijk}$$

$$\forall j: \quad \sum_{i,k} x_{ij} \leq 1$$

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- Middleman buys all goods and sells to buyers, charging according to utility accrued.
  - Given $\underline{p}$, there is a well defined rate for each buyer.
- Equilibrium is captured by a convex program – generalization of EG program.
  - Efficient algorithm for equilibrium
- Market satisfies both welfare theorems.