A New Approach to Strongly Polynomial Linear Programming

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Brief history of LP

- Max c.x   s.t. Ax <= b.

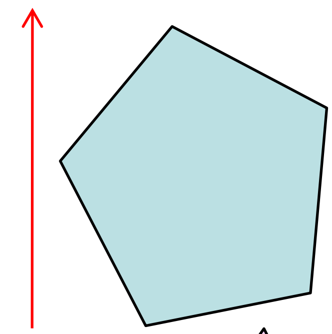
  (several equivalent formulations)

- Simplex method,1947, Dantzig.

  Many variants.

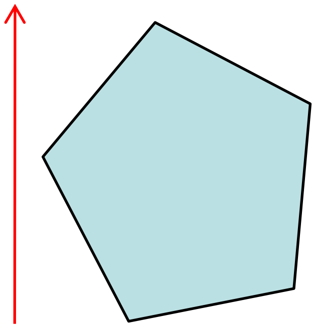
  Still most popular LP algorithm.

  No polynomial guarantee known.



Assume nondegenerate: each vertex is intersection of n facets.

History: polynomial algorithms



- Ellipsoid method, Khachiyan, 1979.
- Interior point method, Karmarkar, 1984.
  - Nestorov, Nemirovski, …
- Perceptron method, Minsky-Papert, 1969
  - Polynomial perceptron, Dunagan-V. 2004
- Random walk method, Bertsimas-V. 2002.

All these methods are geometric.

They “scale” space to make the problem easier; complexity depends on #bits in the input.

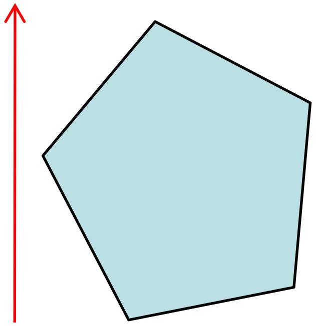
Strongly polynomial cases

- Maximum weight matching in general graphs, Edmonds, 1965.
- Linear programming in fixed dimension, Megiddo, 1984.
- Minimum cost flow,

  E. Tardos, 1986.

These are specialized combinatorial algorithms

Simplex pivot rules



- Assume polyhedron is nondegenerate, i.e., each vertex is the intersection of n facets.
- Then simplex moves from vertex to vertex, improving objective value.
- The rule for determining the next vertex is a “pivot” rule.

Simplex pivot rules

Deterministic pivot rules

- Dantzig’s largest coefficient rule. Exponential example: Klee and Minty, 1972.
- Greatest increase. Example: Jeroslow, 1973.
- Bland’s least index rule. Ex: Avis and Chvatal, 1978.
- Steepest increase. Ex: Goldfarb and Sit, 1979.
- Shadow vertex rule. Ex: Murty, 1980 and Goldfarb, 1983.

- All examples are “variations” of Klee–Minty’s, and fit a general construction called deformed products defined by Amenta and Ziegler.

Simplex pivot rule

- Randomized edge rule: Among all pivots that improve the objective pick one at random.
- Complexity is open.
- Best known upper bound (G. Kalai), is subexponential.
- “Is this pivot rule polynomial?”

- This question has dominated the search for strongly polynomial algorithms.

Hirsch conjecture

- Diameter of a polytope with m facets is at most m-n+1

  (Diameter = Diameter of graph induced by vertices and edges of polytope)

- Best known upper bound is super-polynomial (G. Kalai).
- Long-standing open problem to prove the conjecture or even get a polynomial upper bound.

Hirsch vs Simplex

- Complexity of any simplex pivot rule gives an upper bound on diameter of polytope graph.
- Thus proving randomized simplex conjecture would also be a major combinatorial breakthrough.
- Is strongly polynomial LP really a nongeometric, combinatorial question?

But,

- [Matousek-Szabo] Take a polytope combinatorially equivalent to a hypercube; orient the edges arbitrarily, so that each face has a unique sink. There exist orientations for which the random edge pivot rule is exponential! (pivot rule runs by picking a random out-edge at each vertex visited)

- Does this imply strongly polynomial pivot rules are impossible? NO, because not all orientations are geometrically realizable.
- However, it does suggest that the geometry plays an important role.

New Approach

- Algorithm will be affine-invariant
- So complexity will not depend on how the input is scaled.

Two step iteration:

- At current vertex, pick a line to travel along in an affine-invariant manner and move along the line.
- Go to vertex of at least as high objective value in the face reached.

Step 1: Affine-invariant direction

How to pick a direction?

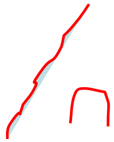
- Compute the set of improving rays (edges that lead to vertices of higher objective value)
- Take a linear combination of the improving rays.

Two candidate rules:

1. Average of all improving rays (centroid rule)
2. Random convex combination of improving rays (random rule).

Lemma: Both rules are affine-invariant.

(i.e. applying an affine transformation before computing the direction gives the same effect as applying it after)



New algorithm, *roughly*

- At current vertex,
  - pick direction
  - Follow to get to new face
  - Go to vertex of at least as high value

- Introduces many geometric shortcuts in the polytope.

- No bearing on Hirsch conjecture!

Step 2: go to vertex

- How to go to a vertex of at least as high value?
  - E.g., Follow gradient, keep adding facets hit as equalities till a vertex is reached.

- Doing this step arbitrarily can lead to an exponential number of iterations.
- So we will go to a vertex in an affine-invariant manner.

- **Algorithm: AFFINE**
- **INPUT:** Polyhedron P given by linear inequalities $a_j . x \leq b_j : j = 1 : m$, objective vector c and vertex z.
- **OUTPUT:** A vertex maximizing the objective value, or “unbounded”

- While the current vertex z is not optimal, repeat:
- 1. (Initialize) (a) Let H be the set of indices of active inequalities at z

  (b) (Compute edges) For every t in H compute a vector

  $v_t : a_h . v_t = 0$ for h in $H\setminus\{t\}$ and $a_t . v_t < 0$.

  (c) Let $T = \{t \text{ in } H : c.v_t \geq 0\}$ and $S = H\setminus T$.

- 2. (Iteration) While T is nonempty, repeat:

  (a) (Compute improving rays) For every t in T compute a vector

  $v_t \neq 0 : a_h.v_t = 0$ for h in $H\setminus\{t\}$, $c.v_t \geq 0$

  and the length of v_t is the largest value for which $z + v_t$ remains feasible.

  (b) (Pick direction) compute a nonnegative combination v of elements of T.

  (c) (Move) Let r be maximal for which $z + r v$ is in P, if there is no such maximum, return “unbounded”. Move the current point: $z := z + r v$.

  (d) (Update inequalities) Let s be the index of an inequality which becomes active. Let t in T be any index such that

  $\{a_h : h \text{ in } \{s\} \cup S \cup T \setminus t\}$ is linearly independent.

  Set $S := S \cup \{s\}$; $T := T\setminus\{t\}$ and $H := S \cup T$.

Algorithm: notes

- In Step 2, one new active inequality is added in each iteration; thus a vertex is reached in at most n iterations.
- Step 2 can be viewed as a recursive application of the original procedure in lower dimensional faces.
- Each iteration: O(mn).

Analysis: How many iterations?

- Klee-Minty: n iterations.
- Main Theorem: For any polytope that is a deformed product, Algorithm Affine takes at most n outer iterations.

  (O(n^2) total iterations).
- Thus, algorithm is efficient on all known simplex counterexamples.

Idea

- Consider P = V x W (standard product)
- Then vertex x of P can be written as
  x = (v,w)
- Step in P projects to step in V or in W. Progress in P goes hand-in-hand with either progress in V or in W.
- Thus f(P) <= f(V) + f(W), where f is the number of vertices visited.

Analysis

- Proof also works for mild perturbations of deformed polytopes, which can change the combinatorial structure considerably (thus algorithm is not specifically designed for these counterexample classes).

- Algorithm makes heavy use of geometric shortcuts through the interior of the polytope.

Next steps

- Polynomial upper bound? (not strongly polynomial, so one could use a geometric scaling type argument showing progress towards optimum).
- Counterexample?
- Upper bound for combinatorial cubes?
- Upper bound for random polytopes? (to get away from the combinatorial structure of known counterexamples)

Thank you!