Adversarial Leakage in Games

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Joint work with
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Two-player zero-sum (binary) games

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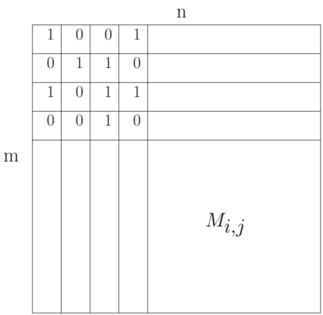

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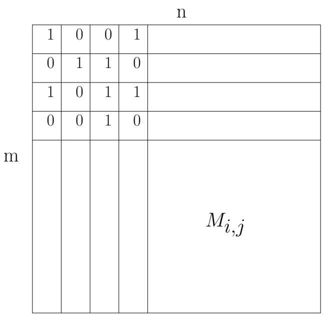

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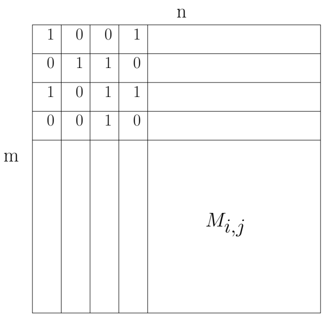

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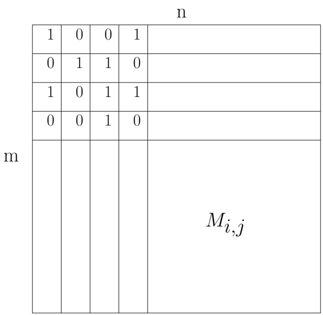

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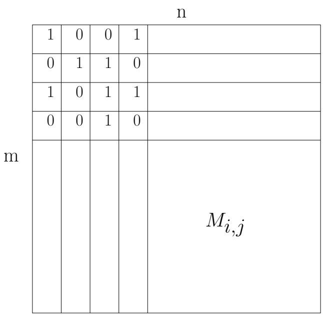

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Theorem (von Neumann, 1928)

$$\max_{p \in \Delta(m)} \min_{j \in [n]} \sum_{i \in [m]} p(i) \cdot M_{i,j} = \min_{q \in \Delta(n)} \max_{i \in [m]} \sum_{j \in [n]} q(j) \cdot M_{i,j} .$$

This is defined to be the value of the game, denoted $\mathrm{v}(M)$ = *the probability that ROW wins.*

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- Question 1:
  what should ROW do now that she knows that COL may learn b bits of information about her pure action?
- Question 2:
  what happens to the value of the game (probability that ROW wins)?

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- Weak model of leakage: COL decides on $f : [m] \rightarrow \{0, 1\}^b$ without knowing the mixed strategy p of ROW.

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ROW wins (step (5)) iff $M_{i,g(f(i))} = 1$.

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The value of the game M under b leaking bits in the strong model:

$$\mathrm{v}^{\mathsf{s}}(M, b) = \max_{p \in \Delta(m)} \min_{f:[m] \rightarrow \{0,1\}^b} \min_{g:\{0,1\}^b \rightarrow [n]} \sum_{i \in [m]} p(i) \cdot M_{i, g(f(i))} \ .$$

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Clearly, $\mathrm{v}^{\mathsf{s}}(M, b) \leq \mathrm{v}^{\mathsf{w}}(M, b) \leq \mathrm{v}(M)$ for every game M and $b \geq 0$.

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If $\mathrm{v}(M) = 1 - \epsilon$ *and* $b \leq \lg(1/\epsilon) - 1$, *then* $\mathrm{v}^s(M, b) \geq 1/2$.

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it is $\geq 1/2$ as long as $b \leq \lg(1/\epsilon) - 1$;
it is < 0.02 once $b \geq \lg(1/\epsilon) + 2$.

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There exists an infinite sequence of games $M \in \{0,1\}^{m \times m}$, $\mathrm{v}(M) = q(1 \pm o(1))$, *so that if* $b \leq \lg \lg(m) - O(1)$, *then* $\mathrm{v}^s(M, b) \geq q^{2^b}(1 \pm o(1))$.

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- $M_{u,v} = 1$ iff the vectors u and v are orthogonal over $GF(2)$.

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Every row and column of M contain exactly $2^{r-1} - 1$ 1-entries.

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- Indeed, requiring that $M_{u,v}=1$ for every $v \in J$ yields a homogeneous system of $|J|$ linear equations over $GF(2)$ in r variables; it has $\geq 2^{r-|J|}-1$ non-zero solutions.
- Playing the uniform distribution on $[m]$ implies the desired $\mathrm{v}^{\mathsf{s}}(M,b) \geq \frac{2^{r-|J|}-1}{m} \geq \left(\frac{1}{2}\right)^{2^b}(1-o(1))$.

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it stays close to $\mathrm{v}(M)$ as long as $b \leq \lg\lg(m) - O(1)$;
it drops to 0 once $b \geq \lg\lg(m) + O(1)$.

Computational complexity

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Theorem

Given a game $M \in \{0,1\}^{m \times n}$ *and some* $b \geq 0$, *both* $\mathrm{v}^s(M,b)$ *and* $\mathrm{v}^w(M,b)$ *are NP-hard to approximate to within any factor.*

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Reducing set cover to the problem of deciding whether the (strong or weak) value is strictly positive.

When b is fixed, computing $\mathrm{v}^s(M,b)$ becomes tractable:

Theorem

Given a game $M \in \{0,1\}^{m \times n}$, the optimal mixed strategy p_b^* can be efficiently computed.

conclusions

Polynomial algorithm for constant b

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Would like to solve the LP

$$\text{maximize } v \text{ s.t.}$$

$$\sum_{i \in [m]} p_i \cdot M_{i,g(f(i))} \geq v \quad \forall f : [m] \to \{0,1\}^b, \forall g : \{0,1\}^b \to [n]$$

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h_J simply maps each row $i \in [m]$ to the column $j \in J$ that minimizes $M_{i,j}$.

Polynomial algorithm for constant b — cont.

Sufficient to solve the LP

$$\text{maximize } v \text{ s.t.}$$

$$\sum_{j \in J} \sum_{i : h_J(i) = j} p_i \cdot M_{i,j} \geq v \quad \forall J \subseteq [n], |J| \leq 2^b$$

$$\sum_{i \in [m]} p_i = 1$$

$$p_i \geq 0 \quad \forall i \in [m] \ .$$

Size = polynomial when b is fixed.

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