A New Approximation Technique for Resource-Allocation Problems

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Relax and Round Paradigm

- **Relaxation**: Given an instance of an optimization problem, enlarge the set of feasible solutions I to some $I' \supset I$.
  - Example: Linear-programming (LP) relaxation of an integer program
- **Rounding**: Efficiently compute an optimum solution $x \in I'$, map x to nearby $y \in I$ and prove that y is near optimum in I

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We will focus on LP-relaxation of 0-1-integer program and LP rounding methods.

LP-Rounding

Various LP-rounding approaches exist.

- Deterministic rounding
- Randomized rounding
    - Independent randomized rounding
    - Dependent randomized rounding

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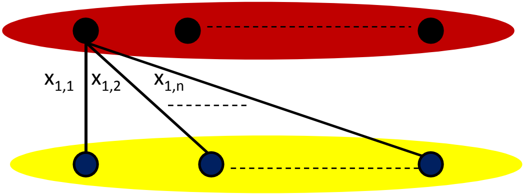
Assignment Problem with Extra Linear Constraints



$$\sum_{j:(i,j)\in E} x_{i,j} = b_i \quad \forall i \in V \qquad \text{(Assign Constraint)}$$

$$\sum_{j:(i,j)\in E} a_{i,j} x_{i,j} \leq T_i \quad \forall i \in V \qquad \text{(Extra Linear Constraints)}$$

Limitation of Independent Rounding

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- Similar kind of problems studied by:
  - General linear constraints: Arora, Frieze & Kaplan [FOCS 96]
  - Constant number of constraints: Papadimitriou & Yannakakis [FOCS 00], Grandoni, Ravi & Singh [ESA 09]

Limitation of Independent Rounding

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- Independent rounding by Raghavan & Thompson.
- By Chernoff-Hoeffding bound, with high probability extra linear constraints are violated by $\pm\tilde{O}(\sqrt{|V|}\max_{i,j} a_{i,j})$.
- Vertices violate matching constraints.

Dependent Rounding

Assignment Problem with Extra Linear Constraints

$$\sum_{j:(i,j)\in E} x_{i,j} = b_i \quad \forall i \in V \qquad \text{(Assign Constraint)}$$

$$\sum_{j:(i,j)\in E} a_{i,j} x_{i,j} \leq T_i \quad \forall i \in V \qquad \text{(Extra Linear Constraints)}$$

- Several works on Dependent Rounding:
  - Ageev & Sviridenko [Journal of Combinatorial Optimization 04]
  - Srinivasan [FOCS 01]
  - Gandhi, Khuller, Parthasarathy, Srinivasan [FOCS 02]
  - Kumar, Marathe, Parthasarathy, Srinivasan[FOCS 05]

Dependent Rounding

Assignment Problem with Extra Linear Constraints

$$\sum_{j:(i,j)\in E} x_{i,j} = b_i \quad \forall i \in V \qquad \text{(Assign Constraint)}$$

$$\sum_{j:(i,j)\in E} a_{i,j} x_{i,j} \leq T_i \quad \forall i \in V \qquad \text{(Extra Linear Constraints)}$$

- Work of Gandhi et al. achieves
  - All matching constraints are satisfied with probability 1.
  - Variables incident on a vertex are negatively correlated.
  - Can still apply Chernoff-Hoeffding bound to get, $\pm\tilde{O}(\sqrt{|V|}\max_{i,j} a_{i,j})$ violation.

Our Rounding Method

Assignment Problem with Extra Linear Constraints

$$\begin{aligned} \sum_{j:(i,j)\in E} x_{i,j} &= b_i \quad \forall i \in V && \text{(Assign Constraint)} \\ \sum_{j:(i,j)\in E} a_{i,j}x_{i,j} &\leq T_i \quad \forall i \in V && \text{(Extra Linear Constraints)} \end{aligned}$$

- Using our rounding method we get,
  - All matching constraints are satisfied with probability 1.
  - Extra linear constraints are violated only by $\pm max_{i,j} a_{i,j}$.
  - Can also handle additional cost constraint: $\sum_{(i,j)\in E} c_{i,j}x_{i,j} \leq C$.

New Rounding Method

Our Rounding Method

$$
\begin{aligned}
\text{minimize} \quad & cx \\
s.t \quad & \\
& Ax \leq b \\
& x \in [0,1]^n
\end{aligned}
$$

- We have an n dimensional system of linear constraints, $Ax \leq b$ with additional constraints, $x \in [0,1]^n$.
- We are given some $x^* \in [0,1]^n$
- We want to round x^* to integer solution.

Our Rounding Method: Main Idea

- Linear constraints define a polytope.

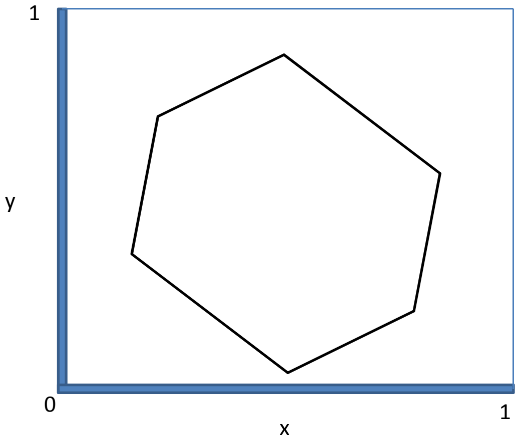
Our Rounding Method: Main Idea

- Not at a vertex of the polytope:
  - Randomly move on the current facet.
    - The expected value of each variable does not change.
  - Either round a new variable or make some other constraint tight.

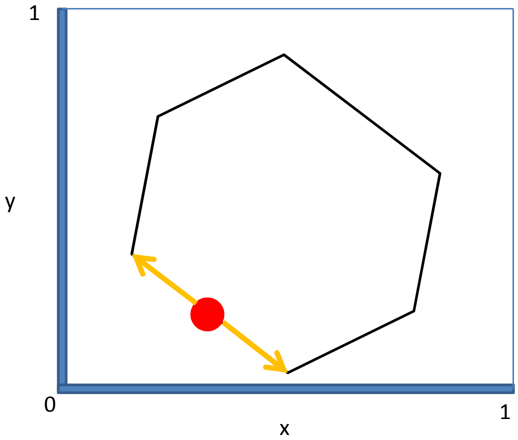
Our Rounding Method: Main Idea

- At a vertex:
  - Relax the polytope by reducing the number of tight constraints.
    - Drop or combine constraints.

Our Rounding Method: Example in 2-dimension



Our Rounding Method: Example in 2-dimension



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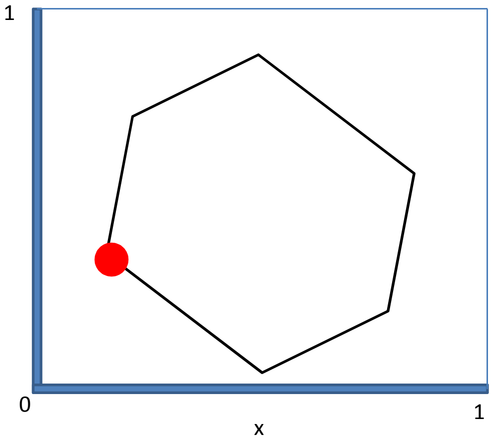

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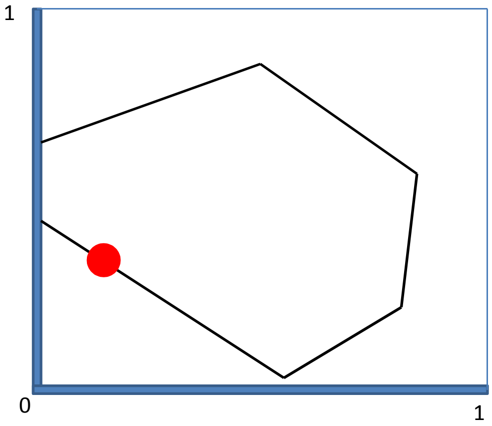

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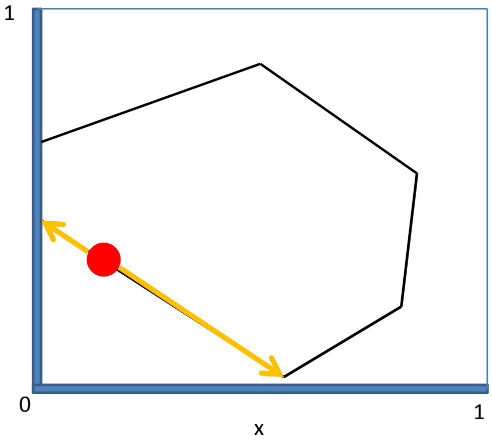

Our Rounding Method

Properties

1. If y^* is the final rounded solution, then $E[y^*] = x^*$
2. A variable rounded to $0, 1$ is never changed.
3. A constraint dropped or combined is never reinstated.
4. At each rounding step, either a new variable gets rounded or a new constraint becomes tight.

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Property (2), (3) and (4) ensure that the process terminates after $O(n + m)$ steps, where n is the total number of variables and m is the total number of constraints.

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4. At each rounding step, either a new variable gets rounded or a new constraint becomes tight.

If no constraint is dropped or combined, then the integer solution obtained is optimum.

Our Rounding Method

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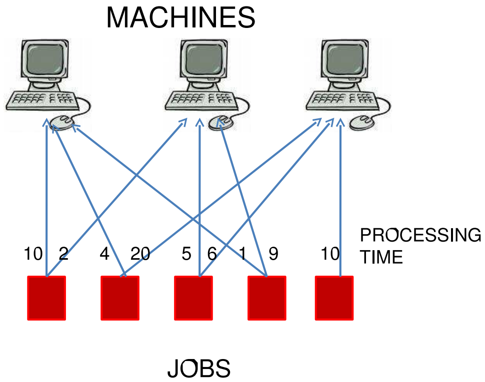
3. A constraint dropped or combined is never reinstated.
4. At each rounding step, either a new variable gets rounded or a new constraint becomes tight.

Choice of constraints to drop or combine is problem specific and are chosen in a way to minimize the violation of the original constraints.

Applications

Unrelated Parallel Machine Scheduling & GAP [8, 9, 10, 1, 5, 7, 6, 2, 3, 4]

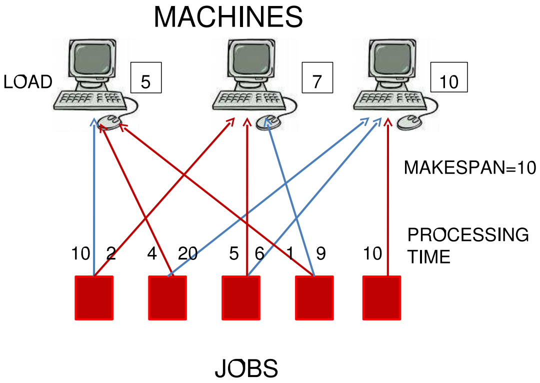


- $p_{i,j}$: processing time of job j on machine i. They are unrelated.

Unrelated Parallel Machine Scheduling & GAP [8, 9, 10, 1, 5, 7, 6, 2, 3, 4]

- **Makespan Minimization**: Minimize the maximum total load (sum of processing time of the allocated jobs) on any machine.

Unrelated Parallel Machine Scheduling & GAP [8, 9, 10, 1, 5, 7, 6, 2, 3, 4]



Unrelated Parallel Machine Scheduling & GAP [8, 9, 10, 1, 5, 7, 6, 2, 3, 4]

- **Makespan Minimization**: Minimize the maximum total load (sum of processing time of the allocated jobs) on any machine.
- 2 approximation for makepspan minimization in UPM by Lenstra, Shmoys & Tardos.

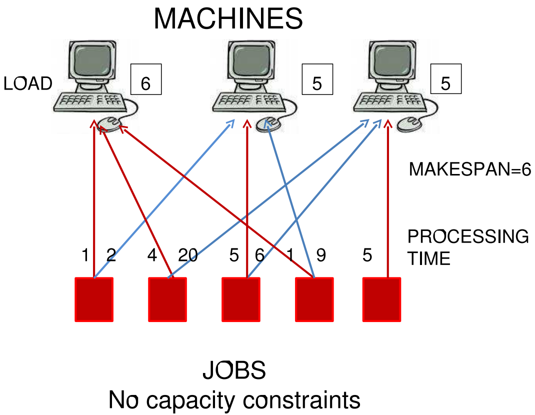
Unrelated Parallel Machine Scheduling & GAP [8, 9, 10, 1, 5, 7, 6, 2, 3, 4]

- **Generalized Assignment Problem (GAP)**: We incur a cost of $c_{i,j}$ if we schedule job j on machine i. Minimize makespan within a cost C.
- (2,1) approximation for makespan and cost for GAP by Shmoys & Tardos.

Applications

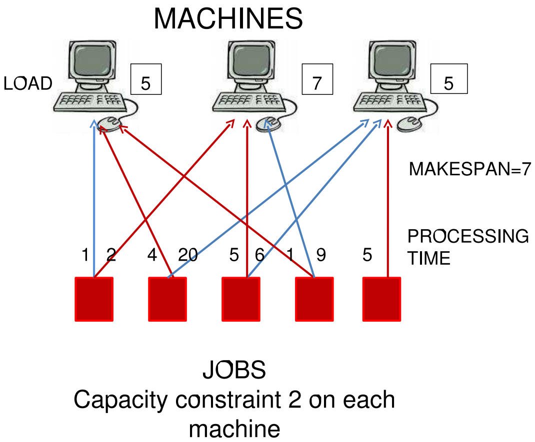
- Extension of unrelated parallel machine scheduling and generalized assignment problem with
  - Hard capacity constraints on machines.
  - Hard profit constraints with outliers.

Applications



- GAP with hard capacity constraint on machines

Applications



- GAP with hard capacity constraint on machines

Applications

- GAP with hard capacity constraint on machines
  - Handling hard capacity constraints is often tricky.
    - Capacitated covering problems, capacitated facility location problem

Applications

- GAP with hard capacity constraint on machines
  - Servers often have limits on the number of jobs they can process.
  - Studied by References.

Applications

- GAP with hard capacity constraint on machines
  - Previously known: 3 approximation to makespan for *identical machines* without any cost constraint.
  - Our result: $(2, 1)$ approximation for GAP with hard capacities.

Applications

- GAP with outliers and hard profit constraints
  - Some jobs may be dropped.
  - Profit associated for scheduling a job.
    - Total profit of the scheduled jobs must be at least Π.

Applications

- GAP with outliers and hard profit constraints
  - Studied by Gupta et al. in APPROX 09.
  - If the optimum makespan is T with profit Π and cost C, the best known approximation bound was $(\Pi, 3T, (1+\epsilon)C)$.
  - We improve it to $(\Pi, (2+\epsilon)T, (1+\epsilon)C)$, for any constant $\epsilon > 0$.

Applications:Others

- Max-min fair allocation problem
  - Closing the integrality gap for a configuration LP considered by Bansal & Sviridenko, Asadpour & Saberi.
  - Extension to equitable partitioning of items.

Applications:Others

- Random bipartite $(b-)$matching with sharp tail bounds for given linear functions.
  - Better approximation factor for special kind of linear functions.

Applications:Others

- Overlay network design.
  - Studied by Andreev, Maggs, Meyerson & Sitaraman.
  - Better approximation factor.

GAP with hard capacity constraints on machines

- Guess the optimum makespan T.

$$\sum_{i,j} c_{i,j} x_{i,j} \leq C \qquad \text{(Cost)} \qquad \sum_{i,j} x_{i,j} = 1 \ \forall j \qquad \text{(Assign)}$$

$$\sum_{j} p_{i,j} x_{i,j} \leq T \ \forall i \qquad \text{(Load)} \qquad \sum_{j} x_{i,j} \leq b_i \ \forall i \qquad \text{(Capacity)}$$

$$x_{i,j} \in \{0,1\} \ \forall i,j$$

$$x_{i,j} = 0 \quad \text{if } p_{i,j} > T$$

- Relax the constraint "$x_{i,j} \in \{0,1\} \ \forall (i,j)$" to "$x_{i,j} \in [0,1] \ \forall (i,j)$" to obtain the LP relaxation.
- Solve the LP to obtain the optimum LP solution x^*

GAP with hard capacity constraints on machines

$$\sum_{i,j} c_{i,j} x_{i,j} \leq C \qquad \text{(Cost)} \qquad \sum_{i,j} x_{i,j} = 1 \ \forall j \qquad \text{(Assign)}$$

$$\sum_{j} p_{i,j} x_{i,j} \leq T \ \forall i \qquad \text{(Load)} \qquad \sum_{j} x_{i,j} \leq b_i \ \forall i \qquad \text{(Capacity)}$$

$$x_{i,j} \in \{0,1\} \ \forall i,j$$

$$x_{i,j} = 0 \quad \text{if } p_{i,j} > T$$

- Ignore (Cost) constraint.
- The expected value of the cost remains same.

GAP with hard capacity constraints on machines

- M_k: Set of all machines with k jobs fractionally assigned to it.

(D1) for each $i \in M_1$, we drop its load and capacity constraints.

(D2) for each $i \in M_2$, we drop its load constraint and rewrite its capacity constraint as $x_{i,j_1} + x_{i,j_2} \leq \lceil x_{i,j_1} + x_{i,j_2} \rceil$, where j_1, j_2 are the two jobs fractionally assigned to i.

(D3) for each $i \in M_3$ for which *both* its load and capacity constraints are tight, drop its load constraint.

GAP with hard capacity constraints on machines

Proof Steps

- Show that the algorithm never reaches a vertex of the polytope. Thus we always make progress.
- Show that dropping constraints **(D1)**, **(D2)** and **(D3)** does not affect capacity and violates makespan only by a factor of 2. ▸ proof
- If the final rounded vector is y^*, then $E[y^*] = x^*$. Thus the expected cost remains C.
  - Can be derandomized directly by the method of conditional expectation to get a cost bound of C.

▸ Skip

GAP with hard capacity constraints on machines

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GAP with hard capacity constraints on machines

In no iteration a vertex of the current polytope is reached.

Skip

GAP with hard capacity constraints on machines

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Notations

- $m_k = |M_k|$, the number of machines with k jobs fractionally scheduled on it.
- n'= the remaining number of jobs that are yet to be assigned permanently to a machine.
- v = the number of variables $x_{i,j} \in (0, 1)$.
- t= the number of linearly independent tight constraints in the current polytope

Skip

GAP with hard capacity constraints on machines

In no iteration a vertex of the current polytope is reached.

Lower and upper bound on v

- $v = m_1 + 2m_2 + 3m_3 + 4m_4 + \ldots$
- $v \geq 2n'$

Hence averaging,

$$v \geq n' + \frac{m_1}{2} + m_2 + \frac{3}{2}m_3 + 2m_4 + \ldots$$

Skip

GAP with hard capacity constraints on machines

In no iteration a vertex of the current polytope is reached.

Number of constraints

- (Assign) constraints: n'
- Tight (Load) and (Capacity) constraints: by our "dropping constraints" steps (**D1**), **D2**) and (**D3**), the number of tight constraints ("Load" and/or "Capacity") contributed by the machines is at most $m_2 + m_3 + \sum_{k \geq 4} 2m_k$.

Hence the total number of constraints
$t \leq n' + m_2 + m_3 + \sum_{k \geq 4} 2m_k$.

Skip

GAP with hard capacity constraints on machines

In no iteration a vertex of the current polytope is reached.

Compare number of variables and constraints

- $v \geq n' + \frac{m_1}{2} + m_2 + \frac{3}{2}m_3 + 2m_4 + \frac{5}{2}m_5 + \ldots$
- $t \leq n' + m_2 + m_3 + 2m_4 + 2m_5 + \ldots$

For the system to be determined, $v \leq t$.

$$m_1 = 0, m_3 = 0, m_5 = m_6 = m_7 = \ldots = 0$$

$$t = v$$

Skip

GAP with hard capacity constraints on machines

In no iteration a vertex of the current polytope is reached.

- Remaining machines have all degree 2 or 4.
- All tight (Assign) and (Capacity) constraints are counted in t.
- But they are not all linearly independent.

Skip

GAP with hard capacity constraints on machines

Capacity constraints are not violated.

- Capacities are integers.
- Capacity constraint is dropped only for machines in M_1.
- Capacity of those machines must be ≥ 1.

Skip

GAP with hard capacity constraints on machines

Makespan is violated at most by a factor of 2

Let X denote the final rounded variable. Then

$$\forall i, \ \sum_{j \in J} X_{i,j} p_{i,j} < \sum_{j} x^*_{i,j} p_{i,j} + \max_{j \in J : x^*_{i,j} \in (0,1)} p_{i,j} \leq 2T$$

- Load constraint is dropped for machines in M_1. But only one extra job (already fractionally assigned to it) can get permanently scheduled on it.

Skip

GAP with hard capacity constraints on machines

Makespan is violated at most by a factor of 2

Let X denote the final rounded variable. Then

$$\forall i, \quad \sum_{j \in J} X_{i,j} p_{i,j} < \sum_{j} x^*_{i,j} p_{i,j} + \max_{j \in J : x^*_{i,j} \in (0,1)} p_{i,j} \leq 2T$$

- Load constraint is dropped for machines in M_2.
  - Capacity$\in (0,1]$: at most one job can be assigned.
  - Capacity$(1,2]$:to start with total fractional assignment is more than 1 and finally all 2 jobs can get permanently assigned to it.

Skip

GAP with hard capacity constraints on machines

Makespan is violated at most by a factor of 2

Let X denote the final rounded variable. Then

$$\forall i, \quad \sum_{j \in J} X_{i,j} p_{i,j} < \sum_{j} x^*_{i,j} p_{i,j} + \max_{j \in J : x^*_{i,j} \in (0,1)} p_{i,j} \leq 2T$$

- Load constraint is dropped for machines in M_3, when they have tight capacity constraints.
  - Capacity of any such machine i must be 1 or 2.
  - Capacity$=$ 1: argued as above.
  - Capacity$=$ 2: to start with total fractional assignment of the jobs was 2. Finally all 3 jobs can get permanently assigned to it.

Skip

Future Direction

- Possible connection with iterative rounding and extensions
- Connection to discrepancy theory ▸ Details.

Thank You!

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BACK.

- Lattice approximation problem: given $A \in \{0,1\}^{m \times n}$ and $p \in [0,1]^n$, obtain a $q \in \{0,1\}^n$ such that $||A.(q-p)||_\infty$ is small.
- $\text{lindisc}(A) = \max_{p \in [0,1]^n} \min_{q \in \{0,1\}^n} ||A(q-p)||_\infty$
- Several results for bounding $\text{lindisc}(A)$ for different matrices.
- Our approach can be potentially useful in bounding the lindisc of random column-sparse matrices.

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