Circuit Lower Bounds, Help Functions, and the Remote Point Problem

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Outline

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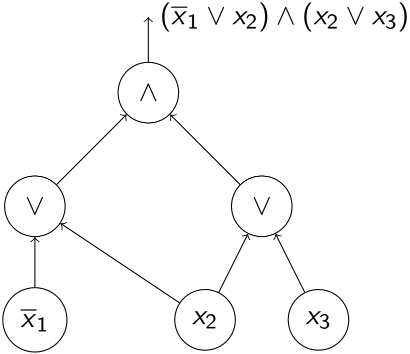
1. Boolean circuits and the Help Functions problem
   - The Help functions problem
   - An application to standard questions
   - The Remote Point Problem (RPP)
   - The connection to the RPP

2. Algebraic Branching Programs with Help polynomials
   - Noncommutative Algebraic Branching Programs
   - Towards explicit lower bounds
   - Results

3. Summary

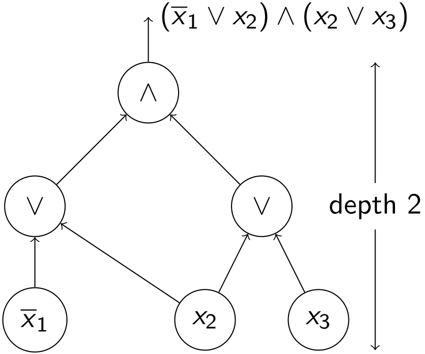
Boolean circuits

- Set of variables $X = \{x_1, x_2, \ldots, x_n\}$.
- Directed acyclic graph (DAG) with labels from $X \cup \overline{X} \cup \{\wedge, \vee\} \cup \{0, 1\}$.
- Computes a function $f : \{0, 1\}^n \rightarrow \{0, 1\}$.



Boolean circuits – parameters

- Size of a circuit – number of vertices.
- Depth of a circuit – The length of the longest path in the circuit.
- Circuits of interest: Constant depth circuits of small size.



Boolean circuit lower bounds

- Notation: Size($s(n)$) – families of functions $\{f_n : \{0,1\}^n \rightarrow \{0,1\}\}_{n \in \mathbb{N}}$ that can be computed by circuits of size $s(n)$. Similarly SizeDepth($s(n), d(n)$).
- $\mathrm{AC}^0 = \mathrm{SizeDepth}(n^{O(1)}, O(1))$.
- AIM: To come up with an explicit (say, computable in EXP) family of boolean functions that cannot be computed by subexponential-sized boolean circuits.
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Boolean circuit lower bounds (contd.)

- Better lower bounds for restricted classes of circuits.
  - Monotone boolean circuits (Razborov, Alon-Boppana): $2^{n^{\Omega(1)}}$ lower bound for CLIQUE.
  - Constant-depth circuits (Furst-Saxe-Sipser, Yao, Håstad): Parity $\notin$ SizeDepth$(2^{n^{\Omega(1)}}, O(1))$.
  - Constant-depth circuits with Mod_p gates and a few Majority gates (Razborov, Smolensky, Aspnes-Beigel-Furst-Rudich) ...
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The Help functions problem

- Fix $h_1, h_2, \ldots, h_m : \{0,1\}^n \to \{0,1\}$ ($m \approx n^{O(1)}$ or $2^{(\log n)^{O(1)}}$).
- What can constant-depth circuits do when given the ability to compute $H = \{h_1, h_2, \ldots, h_m\}$ (on the given input) for "free"?
- Example: Consider constant-depth boolean circuits that, along with $x_1, x_2, \ldots, x_n$, are also given $\bigoplus_{i=1}^n x_i$ as input. Can they compute $\bigoplus_{i \le n/2} x_i$?

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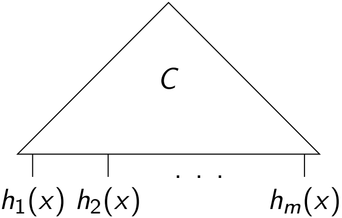
$\text{SizeDepth}_H(s,d)$ - functions computable by circuits of size s and depth d that take functions from H as input.



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- Has been studied by Jin-Yi Cai (1991) and Satya Lokam (1995).
- Cai proves "almost-explicit" lower bounds when $H = \{x_1, \ldots, x_n\} \cup \{h_1, h_2, \ldots, h_k\}$, and $k \leq n^{1/5-\varepsilon}$.
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- Suspected: EXP $\not\subseteq$ Size$(n^{O(1)})$.
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R_2

R_3

$\vdots$

$\boxed{x}$ R_n

$|x| = n$

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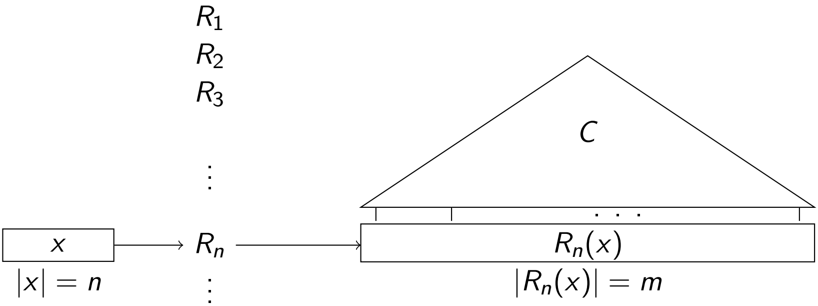

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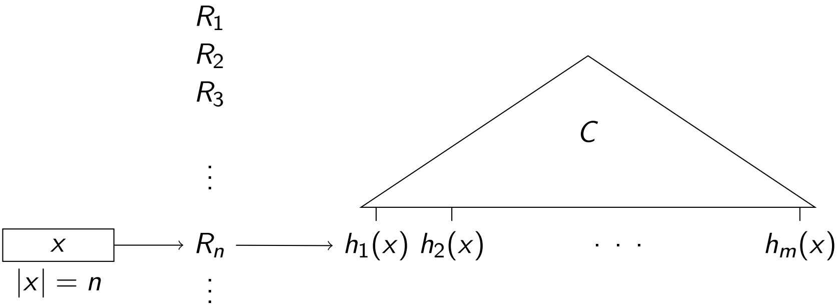

Our observation

A solution to the Help Function problem (for constant-depth circuits) would follow from a "good" solution to the Remote Point Problem.

The Remote Point Problem (RPP)

- Define the $(k(N), r(N))$-Remote Point Problem (RPP) as follows:
  - INPUT: A basis for a subspace V of $\mathbb{F}_2^N$ of dimension at most $k = k(N)$.
  - SOLUTION: A vector $u \in \mathbb{F}_2^N$ such that $\Delta(u, v) \geq r(N)$ for all $v \in V$.
- Here, $\Delta(x, y)$ is the Hamming distance between x and y: that is, $|\{i \in [n] \mid x_i \neq y_i\}|$.

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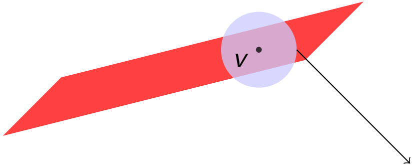

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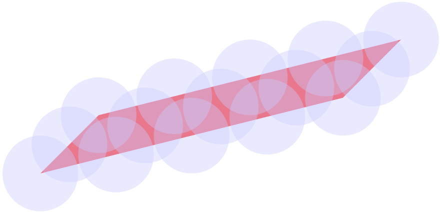

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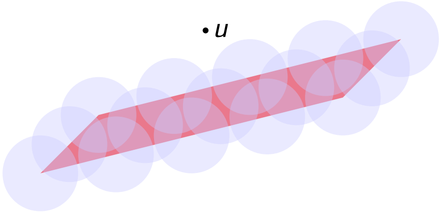

Motivation and previous work

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- An interesting "restriction" of the Matrix Rigidity question.
- The Matrix Rigidity question may be phrased in terms of small hitting sets for the RPP.
- Interesting parameters: $(k(N) = N/10, r(N) = N/10)$. Random point is a solution w.h.p.. Need a deterministic solution.
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- C - small constant-depth boolean circuit with m inputs.
- Using low-degree polynomial approximations to AC^0 (Razborov, Smolensky, Tarui), there is a polynomial p_0 of small degree (at most $\ell = \log^{O(1)}(m)$) such that,

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The connection to the Help functions problem (contd.)

0	1	0	0	1	· · · ·	0	1	0	$C(h_1(x), \ldots, h_m(x))$

Hamming distance $< \varepsilon 2^n$.

0	0	0	0	1	· · · ·	1	1	0	$p_0(h_1(x), \ldots, h_m(x))$

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Does this help?

- Does the connection to the RPP give us a non-trivial solution to the Help functions problem?
- Not really. The best solution currently (Alon et. al.) is a $(k, N^{\frac{\log k}{k}})$-solution. Need a $(k, N^{\frac{1}{k^{o(1)}}})$-solution.
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Outline

Noncommutative Algebraic Branching Programs (ABPs)

- Field $\mathbb{F}$. Set of variables $X = \{x_1, x_2, \ldots, x_n\}$.
- Noncommutative ring of polynomials $\mathbb{F}\langle X \rangle$. $x_1 x_2 \neq x_2 x_1$.

The RMP

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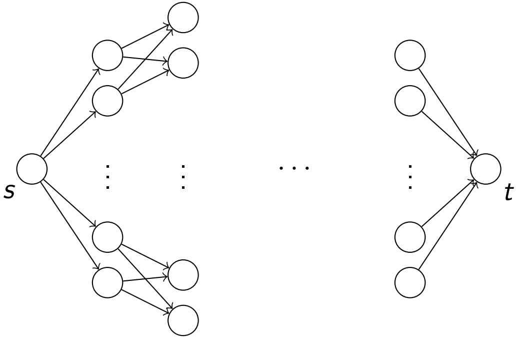

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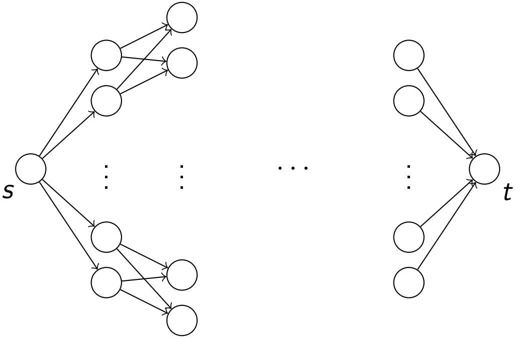

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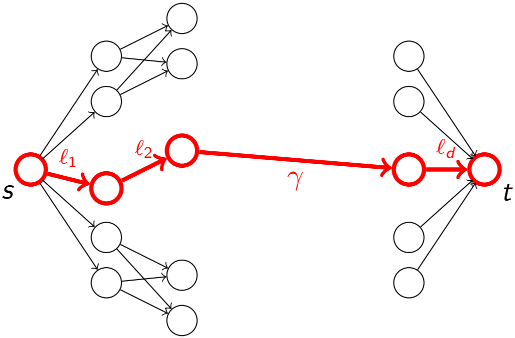
$$f_\gamma = \ell_1 \ell_2 \cdots \ell_d$$



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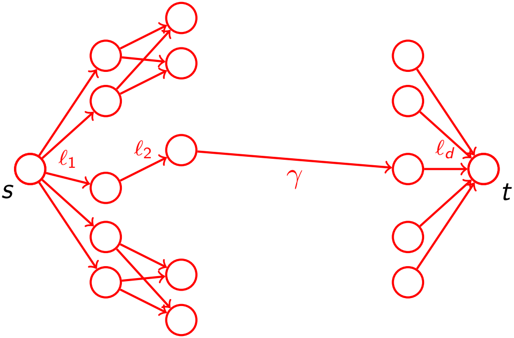

$$f_\gamma = \ell_1 \ell_2 \cdots \ell_d$$

$$f = \sum_{\gamma \in \mathcal{P}_{st}} f_\gamma$$

The RMP

Properties

- An ABP with d layers computes homogeneous (degree d) polynomials in the noncommutative ring $\mathbb{F}\langle X \rangle$.
- Size of an ABP A: the number of vertices in the underlying graph.
- ABPs at least as powerful as arithmetic formulas.
- Nisan proved exponential lower bounds for the size of ABPs computing a whole range of *noncommutative* polynomials, such as the Determinant, the Permanent, etc.
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Noncommutative ABPs with help polynomials

- Fix $H = \{h_1, h_2, \ldots, h_m\}$, a set of arbitrary polynomials from the noncommutative ring $\mathbb{F}\langle X \rangle$.
- ABPs with help polynomials H - Same as standard ABPs, except we allow the h_i in the linear forms.

$$\ell = \sum_i \alpha_i x_i + \sum_j \beta_j h_j$$

- The ABP with help polynomials lower bound question: Given $H = \{h_1, h_2, \ldots, h_m\}$, compute a polynomial F such that F cannot be computed by a small ABP using H.

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The communication matrix $M_k(f)$

- Fix $f \in \mathbb{F}\langle X\rangle$ homogeneous of degree d.
- $\mathrm{Mon}_\ell(X)$ – monic monomials of degree ℓ.
- $f(m)$ – coefficient of monomial m in f.
- For $0 \le k \le d$, the matrix $M_k(f)$ is an $n^k \times n^{d-k}$ matrix over $\mathbb{F}$ such that:
  - The rows are labelled by elements of $\mathrm{Mon}_k(X)$.
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  - The (m_1, m_2)th entry is $f(m_1 m_2)$.

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$$
\begin{array}{c c}
 & \mathrm{Mon}_{d-k}(X) \longrightarrow \qquad m_2 \\
\begin{array}{c} \mathrm{Mon}_k(X) \\ \downarrow \\ \\ \\ m_1 \\ \\ \\ \end{array} &
\begin{pmatrix}
. & . & . & . & & \cdots & . \\
. & . & . & . & & \cdots & . \\
. & . & . & . & & \cdots & . \\
 & & & & f(m_1 m_2) & & \\
\vdots & \vdots & \vdots & \vdots & & \cdots & . \\
. & . & . & . & & \cdots & .
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\end{array}
$$

The approach to lower bounds

- Say we have a small ABP A computing f using H.
- Then, $M_{d/2}(f) = M' + M$, where:
  - M' small rank.
  - $M \in V(H)$, where $V(H)$ a small dimensional vector space depending *only* on H.
- Thus, for an explicit lower bound, it suffices to find M_0 such that $\mathrm{rank}(M_0 - M)$ is large for *every* $M \in V(H)$. Then, choose $F \in \mathbb{F}\langle X\rangle$ so that:

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The Remote Matrix Problem (the RPP with rank metric)

- Let $\Delta_{\text{rank}}(M_1, M_2) = \text{rank}(M_1 - M_2)$.
- The $(k(N), r(N))$-Remote Matrix Problem (RMP) is defined as follows:
  - INPUT: A collection of matrices $M_1, M_2, \ldots, M_k \in \mathbb{F}^{N \times N}$.
  - SOLUTION: A matrix $M \in \mathbb{F}^{N \times N}$ such that $\Delta_{\text{rank}}(M - M') \geq r$ for each $M' \in \text{span}(M_1, M_2, \ldots, M_k)$.
- Easy parameters: The $(k, N/(k+1))$-RMP has an easy solution.
- Interesting parameters: $k = N^2/10$, $r = N/10$. Random point is a solution w.h.p..

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Lemma

The $(k, N/(k+1))$-RMP can be solved in polynomial time.

Theorem

There is an explicit lower bound F against ABPs using H if:

- *H is not too large.*
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If the $(k, N/k^{1/2-\varepsilon})$-RMP can be solved in polynomial time, then there is an explicit lower bound F against ABPs using H, for any H that is not too large.

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Other Results

Following the general proof structure of the result of Alon, Panigrahy, and Yekhanin's result on the RPP:

Theorem

The $(N, \log N)$-RMP can be solved in polynomial time, for constant-sized fields.

Outline

Summary

- We studied the computational model of constant-depth boolean circuits with help functions, and Noncommutative ABPs with help polynomials.
- We showed connections between the Help function problem and the problem of separating EXP from the polynomial-time many-one closure of SizeDepth($n^{O(1)}, O(1)$).
- We also showed connections between the Help function/polynomial problems and solving the Remote Point Problem in the Hamming and rank metrics respectively.
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Open questions

- Algorithms with better parameters for the RPP and RMP.
- Specific cases of the Help functions question:
  - Is there a small H such that $\mathrm{SizeDepth}_H(n^{O(1)}, O(1))$ contains all the parities?
  - If H contains only parities, then does $\mathrm{SizeDepth}_H(n^{O(1)}, O(1))$ contain the inner-product function?
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Thank you