Bounds on the quantum satisfiability threshold

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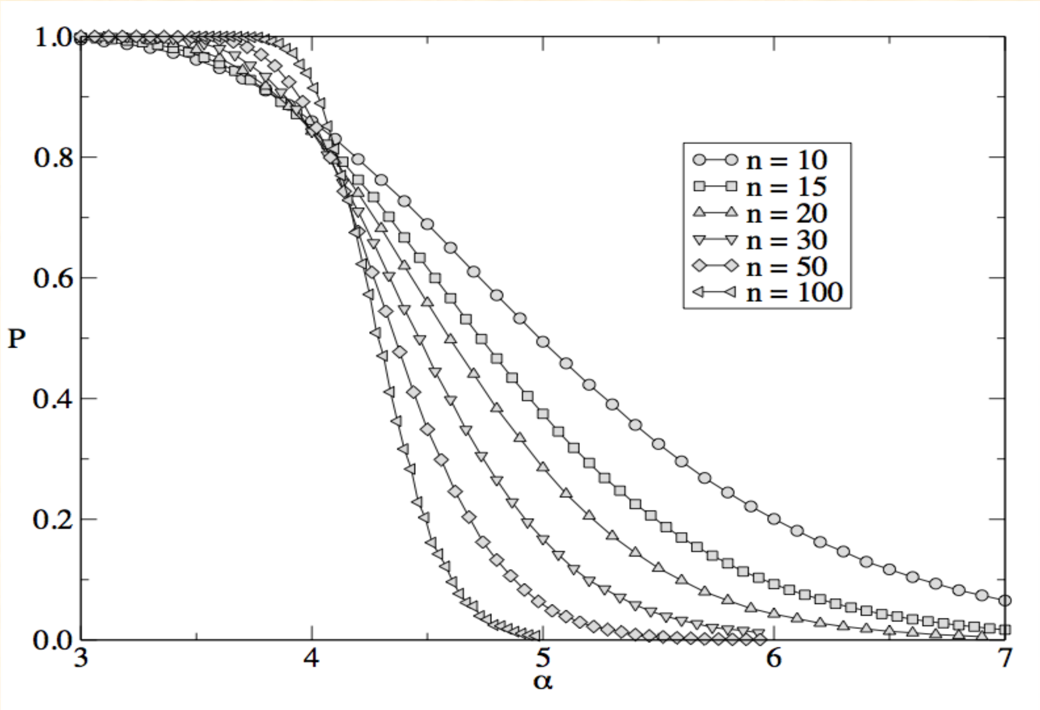
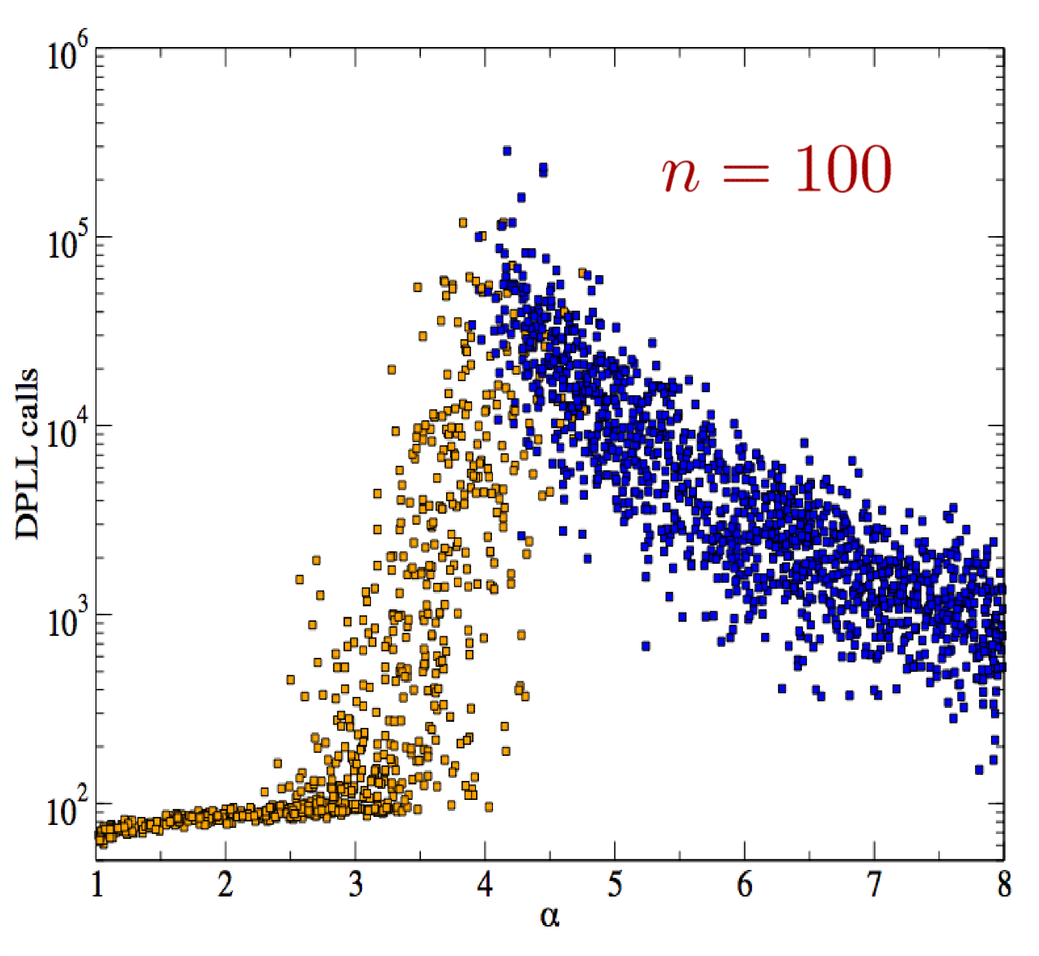
ICS 2010,
Tsinghua University, Beijing
January 7, 2010

Random 3-SAT formulas with n variables and $m = \alpha n$ clauses exhibit a phase transition for $\alpha = \alpha_c \approx 4.267$

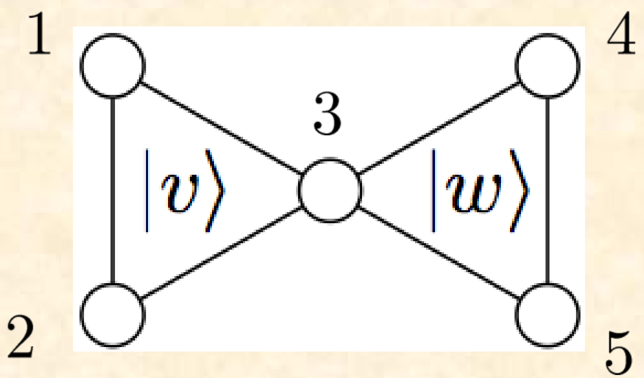


$$\lim_{n\to\infty} \Pr[\phi(n, \alpha n) \text{ is satisfiable}] = \begin{cases} 1 & \text{if } \alpha < \alpha_c \\ 0 & \text{if } \alpha > \alpha_c \end{cases}$$

Search times appear to peak at the transition point



classical	quantum
n-bit assignment $x \in \{0,1\}^n$	A unit vector $\in (\mathbb{C}^2)^{\otimes n}$ $\lvert\psi\rangle = \sum_{x \in \{0,1\}^n} a_x \lvert x\rangle$
3-bit clause $(x_1 \vee \neg x_2 \vee x_3)$ forbidden substring 010	Forbidden vector $\lvert v\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ for some triple of qubits
Assignments violating a clause $\lvert 010\rangle \otimes \lvert 00\ldots0\rangle$ $\lvert 010\rangle \otimes \lvert 00\ldots1\rangle$ $\lvert 010\rangle \otimes \lvert 11\ldots1\rangle$	Forbidden subspace $\text{span}\left\{\begin{array}{c} \lvert v\rangle \otimes \lvert 00\ldots0\rangle \\ \lvert v\rangle \otimes \lvert 00\ldots1\rangle \\ \ldots \\ \lvert v\rangle \otimes \lvert 11\ldots1\rangle \end{array}\right\}$



$$V_{\text{forbidden}} = \text{span}\left\{ \begin{array}{c} |v\rangle \otimes |00\rangle, \\ |v\rangle \otimes |01\rangle, \\ |v\rangle \otimes |10\rangle, \\ |v\rangle \otimes |11\rangle, \\ |00\rangle \otimes |w\rangle, \\ |01\rangle \otimes |w\rangle, \\ |10\rangle \otimes |w\rangle, \\ |11\rangle \otimes |w\rangle, \end{array} \right\}$$

The orthogonal complement to the forbidden subspace is called the satisfying subspace V_{sat}. It is spanned by n-qubit states orthogonal to every forbidden vector.

The satisfying rank $R_{\text{sat}} = \dim(V_{\text{sat}})$ is analogous to the number of satisfying assignment.

A formula is satisfiable iff $R_{\text{sat}} \geq 1$

A quantum k-SAT formula:

- A hypergraph of clauses G. Vertices of G are qubits $1, 2, \ldots, n$. Edges of G are k-tuples of qubits.
- A choice of a k-qubit forbidden vector $|v_i\rangle$ for every edge i.

Worst-case complexity of quantum k-SAT:

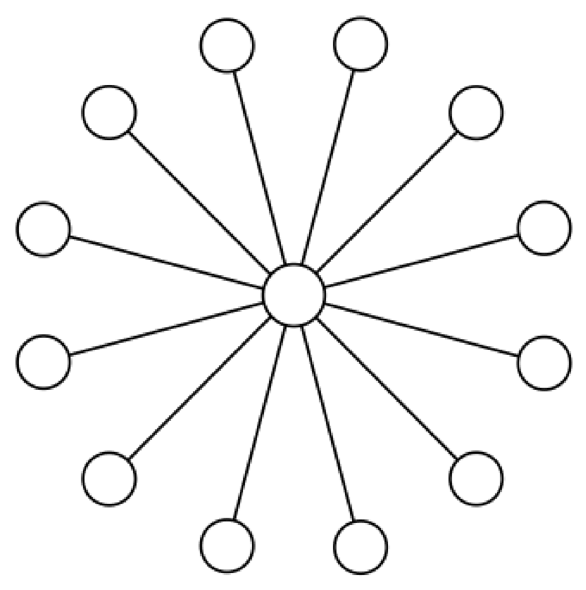
$k = 2$: solvable in time $poly(n)$

$k \geq 4$: complete for QMA_1 (quantum analogue of Merlin-Arthur games with perfect completeness)

$k = 3$: somewhere between NP and QMA_1

S.B. quant-ph/0602108

Quantum k-SAT is more restrictive



2-SAT on a star of degree d

Classical: at least $2^{d/2}$ solutions

Forbid singlets $|v\rangle \sim |0,1\rangle - |1,0\rangle$

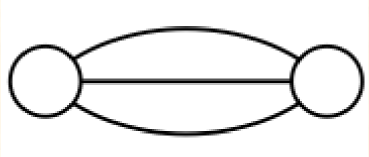
Satisfying rank $R_{\text{sat}} = d + 2$

Any state orthogonal to a singlet is symmetric the transposition of the two qubits

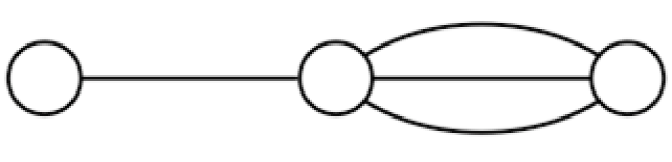
If the graph of clauses is connected, satisfying states must be symmetric under all permutations of qubits

Quantum k-SAT is more restrictive

This 2-SAT formula is satisfiable:



Is this one?



Classical: of course! Use the new variable to satisfy the new clause

Quantum: no! In entangled states single variables don't have values. Similarly, a single variable cannot satisfy entangled clause.

Random quantum k-SAT formulas

(Laumann et al 2009)

Two sources of randomness:

(1) Random hypergrath with n vertices and $m = \alpha n$ edges. Edges are chosen uniformly with replacement

(2) Random forbidden vectors $|v\rangle \in (\mathbb{C}^2)^{\otimes k}$ chosen uniformly from the set of unit vectors

Threshold conjecture:

$$\lim_{n\to\infty} \Pr[\phi(n,\alpha n) \text{ is satisfiable}] = \begin{cases} 1 & \text{if } \alpha < \alpha_q \\ 0 & \text{if } \alpha > \alpha_q \end{cases}$$

Our contribution: upper bounds on α_q

Generic satisfying rank

Consider first only one source of randomness:

The hypergraph of clauses: fixed

The forbidden vectors: random

The satisfying rank R_{sat} is a random variable with non-negative integer values

Geometrization Lemma (Laumann et al 09):
R_{sat} takes its smallest possible value with probability 1.

This defines generic satisfying rank $R_{\mathrm{sat}}^{\mathrm{gen}}$ for a given hypergraph of clauses

Generic satisfying rank for 2-SAT

Let G be a connected graph of clauses with n vertices

G	generic satisfying rank
a tree	$n+1$
a cycle	2
a tree with a double edge	2
a triple edge	1
all other graphs	0

Open problem: how to compute $R_{\mathrm{sat}}^{\mathrm{gen}}$ for $k \geq 3$?

Threshold conjecture:

$$\lim_{n\to\infty} \Pr[\phi(n,\alpha n) \text{ is satisfiable}] = \begin{cases} 1 & \text{if } \alpha < \alpha_q \\ 0 & \text{if } \alpha > \alpha_q \end{cases}$$

Random hypergraph: $R_{\text{sat}}^{\text{gen}}$ becomes a random variable

$\alpha_q < \alpha$ iff $R_{\text{sat}}^{\text{gen}} = 0$ with high probability

$\alpha_q > \alpha$ iff $R_{\text{sat}}^{\text{gen}} \geq 1$ with high probability

We shall get upper bound on α_q by deriving a simple upper bound on $R_{\text{sat}}^{\text{gen}}$. This bound involves a decomposition of the hypergraph in terms of simpler hypergaphs (gadgets).

Decomposition of the hypergraph into gadgets:

$$G = G_1 \cup G_2 \cup \ldots \cup G_p$$

Each gadget G_i involves one or several clauses acting on some subset of t_i qubits.

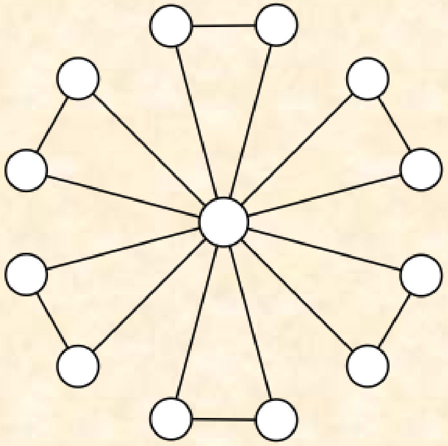
Factorization Lemma

The generic satisfying rank is submultiplicative:

$$R_{\mathrm{sat}}^{\mathrm{gen}}(G) \le 2^n \prod_{i=1}^{p} 2^{-t_i} R_{\mathrm{sat}}^{\mathrm{gen}}(G_i)$$

We need a decomposition of G into gadgets that yields upper bound $R_{\mathrm{sat}}^{\mathrm{gen}} < 1$ with high probability.

The Sunflower



This is $(6,3)$-sunflower

The generic satisfying rank of the (d,k)-sunflower is

$$R_{\rm sat}^{\rm gen} = 2(2^{k-1}-1)^d \left(\frac{d}{2^k-2} + 1 \right)$$

Given a decomposition with n_d sunflowers of degree d one gets

$$R_{\rm sat}^{\rm gen}(G) \leq 2^n \prod_{d=1}^{\infty} \left(\left(\frac{3}{4}\right)^d \left(\frac{d}{6}+1\right) \right)^{n_d} \qquad (k=3)$$

How to partition into sunflowers?

Naive: at each step, choose a random vertex, declare it and its clauses to be a sunflower, and remove them

Continuous time: give each vertex an index $t \in [0,1]$ and remove in the decreasing order

The degree of a sunflower of index t is the number of clauses whose variables all have index $< t$. This is the Poisson distribution with mean $k\alpha t^{k-1}$. It yields

$$\mathbb{E}(n_d) = n \int_0^1 dt \, \mathrm{Poi}(k\alpha t^{k-1}, d)$$

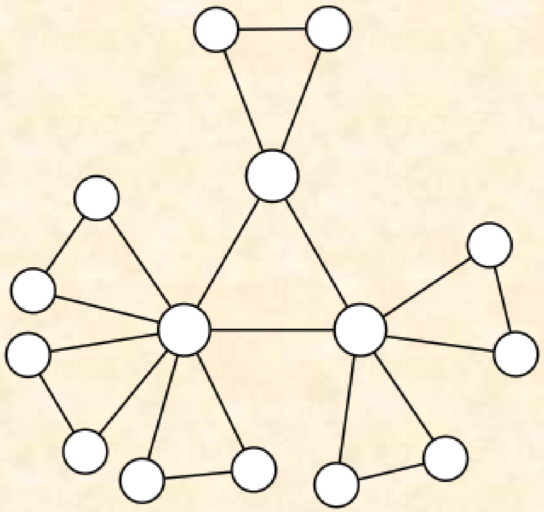
Assuming that $n_d > \mathbb{E}(n_d) - o(n)$ for $1 \leq d \leq 100$ we get exp. small upper bound on $R_{\mathrm{sat}}^{\mathrm{gen}}$ for $\alpha < 3.894$. It implies

$$\alpha_q < 3.894 < \alpha_c$$

Bigger gadgets: more conflicts, smaller rank

The Nosegay

nosegay
noun
a small bunch of flowers,
typically one that is sweet-scented



$$R_{\text{sat}}^{\text{gen}} = 3^{a+b+c-3}\left[(a+6)(b+6)(c+6) - (a+3)(b+3)(c+3)\right]$$

Naive: At each step, choose a random clause, declare it and its neighbors to be a nosegay, and remove them

It gives $\alpha_q \le 3.594$. This is well below the classical threshold $\alpha_c \approx 4.267$

Open questions

Generic satisfiability: given a hypergraph of clauses, decide whether it is satisfiable for generic choice of forbidden vectors. Is this problem in NP ? Is it NP-hard?

Is there a satisfiable-but-entangled phase, in which random formulas are satisfiable, but all satisfying states are highly entangled?

What is the adversarial classical threshold, where the hypergraph is random, but the adversary chooses which literals to negate?