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Revenue Maximization via Nash Implementation

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Abstract: We consider the problem of maximizing revenue in prior-free auctions for general single parameter settings. The setting is modeled by an arbitrary downward-closed set system, which captures many special cases such as single item, digital goods and single-minded combinatorial auctions. We relax the truthfulness requirement by the solution concept of Nash equilibria. Implementation by Nash equilibria is a natural and relevant framework in many applications of computer science, where auctions are run repeatedly and bidders can observe others' strategies, but the auctioneer needs to design a mechanism in advance and cannot use any information on the bidders' private valuations.
We introduce a worst-case revenue benchmark which generalizes the second price of single item auction and the $\mathcal{F}_2$ benchmark, introduced by Goldberg et al., for digital goods. We design a mechanism whose Nash equilibria obtains at least a constant factor of this benchmark and prove that no truthful mechanisms can achieve a constant approximation.

Keywords: mechanism design, single parameter, Nash equilibria.

1 Introduction

Revenue maximization in mechanism design has an extensive history, which primarily begins with with the seminal papers of Myerson [13] and of Riley and Samuelson [17]. These papers study optimal auctions in Bayesian settings, where bidders' valuations are drawn from commonly known distributions and the mechanism designer relies on these distributions. In recent years, a different approach, known as *prior-free mechanism design*, has gained much attention from researchers in theoretical computer science. Prior-free mechanism design aims to overcome the problem that Bayesian auctions highly depend on the distributions of bidders' valuation, which are hard or impossible to obtain in many scenarios. The main challenge in prior-free mechanism design is that there are no characterizations for *the optimal* auction as in the Bayesian settings. Research in this direction has been taking the *revenue benchmark* approach, which has been applied successfully in several settings (see the survey [5]). The idea is to define a function on the valuation vector, called *benchmark*, that presents an upper bound on the revenue of "reasonable" mechanisms. And the goal is to design a mechanism, which achieves a constant fraction of the benchmark. Such mechanism is called a *competitive* mechanism.

The current works in this direction mainly focus on designing competitive *truthful* auctions. Thus far, however, competitive truthful auctions are mostly known for simple auction settings, where the goods are in unlimited supply and/or the bidders are symmetric. This is in part because of the strong condition on truthful auctions, which requires that, it is best for bidders to report their true valuation regardless of what other bidders do. This concern is also expressed in [5]: " Truthfulness may needlessly limit our ability to achieve our goals [maximizing revenue]. ... Thus, one of the most important research directions for future research is to consider alternative solution concepts".

Motivated by this line of research, in this paper, we consider the problem of designing competitive auctions with the solution concept of *Nash equilibria* in full information settings. In contrast with truthful auctions, Nash equilibria only require that no bidder would change his bid if others keep their strategies. In this setting, bidders observe each other's bids and adjust their bids accordingly, but an important constraint is that auctioneers need to design a mechanism with no information on bidders' private valuations.

The setting, usually called Nash Implementation, is a natural and relevant framework in many applications, where auctions are run repeatedly and bidders can observe and adapt to others' bids to optimize their

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payoff. The auctioneer, on the other hand, needs to design a mechanism in advance and has to commit to this mechanism in the future. Therefore, he/she does not have any information on the valuation of bidders to design the mechanism.

In the literature of Nash implementation, most mechanisms require bidders to submit others' private information; if any two bidders report different information, they will get a large penalties. This type of mechanisms is unnatural and impractical.

In this paper, we are interested in a more natural class of mechanisms, where the outcome depends continuously on the bid vector, further more, bidders can only learn and adapt based on others' bids, not on their private valuations. The main question we ask is: With this natural type of mechanisms, and using Nash equilibria as the solution concept, can we overcome the drawbacks of truthful mechanisms? Namely, can we define a natural revenue benchmark and design a competitive mechanism for *general auction settings*, where competitive truthful mechanisms are not known? Can we obtain *more* revenue in this framework than in truthful auctions?

Our results. We consider a general single parameter auction setting, which is also studied in [6]. Here, each bidder has a private valuation for receiving a service and there is a set system representing *feasible sets*. A feasible set is a set of bidders that can be served simultaneously. For example, in single item auctions the feasible set system contains only singletons; in digital good auctions, the set system contains all subsets of the bidders. We focus on the general case of downward-closed environment where every subset of a feasible set is again feasible. An important example of this environment is a combinatorial auction with single-minded bidders, where feasible sets correspond to subsets of bidders seeking disjoint bundles of goods.

We define a natural benchmark and design a mechanism that generates at least 1/14 fraction of this benchmark. We prove that no truthful mechanisms can be competitive against our benchmark.

Our mechanism combines two natural classes of mechanisms: a truthful and the proportional sharing mechanisms. Our mechanism might have multiple equilibria. Therefore, one potential criticism is that among many equilibria, there might be one that gives low revenue. However, this is also an issue for weakly dominant strategy truthful mechanisms[1]. A part of our mechanism uses a truthful mechanism and interestingly, this is the only part that causes the existence of an equilibrium with low revenue. More precisely, if we assume that in weakly dominant strategy truthful mechanisms bidders bid truthfully, or in other words, if bidders do not play dominated strategies, then *every* Nash equilibrium of our mechanism generates at least a constant fraction of the benchmark.

The main idea. The two special cases of our setting are the single item auction, which has a full competition among bidders, and the digital goods auction with no competition among bidders. Our benchmark and mechanism can be seen as a combination of these two extreme cases. The benchmark is the generalization of the second price in single item and the $\mathcal{F}_2$ benchmark, proposed in [4], for digital good. Consider the case where we can partition the set of bidders into two sets N_1 and N_2, N_2 being a set of bidders that can be served simultaneously. From N_1 we can get at most the maximum social welfare of this group of bidders, denoted by $\text{SocialOpt}(N_1)$, for N_2, because of the lack of competition, we will use the benchmark $\mathcal{F}_2(N_2)$. Taking the minimum over all partitions, we can define the following benchmark, which gives a generalization of the two special cases.

$$\mathcal{R} = \min_{N_1,N_2} \text{SocialOpt}(N_1) + \mathcal{F}_2(N_2),$$

where N_1, N_2 is a partition of the bidders and N_2 can be served simultaneously.

To design a competitive mechanism against $\mathcal{R}$, we first observe that the randomized outcomes of our setting can be captured by a polyhedron, for which the proportional sharing mechanism is well understood [8, 14]. Proportional sharing for a single constraint, for example $\sum_i x_i \leq 1$ is a mechanism, such that if bidder i bids b_i, then his allocation x_i is $\frac{b_i}{\sum_k b_k}$ and his payment is b_i. For a general polyhedral setting, each bidder needs to bid on all the constraints of the polyhedron. This is a mechanism that can create competition among bidders. An intuitive way to understand proportional sharing is by a similar game, where we assume that bidders are "price taking" [9]. In this game, there are a set of unit prices on the constraints of the polyhedron, and bidders try to optimize their payoff according to these prices. The prices can be seen as a dual vector of an optimization program. If

[1]A simple example is the case of second price auction for a single item. It is a Nash equilibrium if, except for the highest bidder who bids truthfully, all other bidders bid 0. The revenue is 0 in this case

we know the valuations of the bidders, we can find the prices to get a revenue as high as the optimal social welfare. However, when the prices are the functions of the bid vector, there is a "price shading" effect. Bidders would not increase their bids to match the prices resulted in the price taking game. This problem is more serious with bidders of high valuations and bidders with little competition. We are not able to get high revenue from them.

The idea is that, after the proportional sharing mechanism is run, we can observe the set of bidders who get a large share of resources, which we call *big bidders*. Note that this set of bidders depend on the outcome of the proportional sharing mechanism. If the resource that these bidders get is large enough, we know that they need to belong to a feasible set. One can say that there is a lack of competition among them. We will give additional resources to these bidders for extra money by using the mechanism designed for digital goods.

This natural idea of attaching a truthful mechanism after a proportional sharing mechanism, however, has several issues. Because the second mechanism is run on the outcome of the first mechanism, bidders might behave differently from the case where the two mechanism are run separately. There are two main issues. First, the second phase of the mechanism is run only for the set of big bidders who get large share of the resource in the first round, therefore, it might be the case that the *small bidders* will overbid in the first round to get to the second one. Thus, the property of a Nash equilibrium in proportional sharing might be not valid. Second, it is also possible that the large bidders will change the equilibrium bid of the game in the first round to change the set of bidders that survive to the second round, and thus the price of the second mechanism might be better for them.

To overcome these difficulties, we need to modify both mechanisms. For the proportional sharing, in particular, we introduce a "truncated proportional sharing" mechanism, which sets an upper bound on allocations of bidders in the first round to guarantee that big bidders cannot benefit from over bidding to change the set of bidders that survive to the second round.

Related work. Profit maximization in mechanism design was first studied in the seminal papers of Myerson [13] and of Riley and Samuelson [17]. These papers characterize optimal auctions in Bayesian settings, which is by now standard and can be found in basic texts on auction theory. Design of prior-free mechanisms is an important topic of computer science literature. The approach was first considered by [4] and has a large literature, see, for example, the survey [5] and the citations therein.

The main distinction in our approach is to use Nash equilibria in full information setting as the solution concept. This approach belongs to the theory of Nash implementation in economics literature. In this framework it is assumed that agents (bidders in our case) have full information about each other's preferences, but the planner (auctioneer) only knows the set of outcomes and does not have any information on the private types of agents. The literature was initiated with the seminal work Maskin [10], where a characterization of the set of implementable outcomes is given. There is a large literature on this topic see the surveys [10, 11, 16].

The type of mechanisms used in Nash implementation is, however, not natural. It is usually required that bidders need to report others' preferences (private valuations), and they will get some large penalties if any two bidders do not report the same information. The mechanism we use in this paper is the combination of two natural, well known classes of mechanisms: proportional sharing and a weakly dominant strategy truthful mechanism.

Our paper is not the first attempt in algorithmic mechanism design literature to relax the dominant strategy solution concept. Babaioff, Lavi and Pavlov [2] consider the concept of implementation in undominated strategies. The focus of their work is the computational issues and social welfare of single value combinatorial auctions. Our setting and mechanism can be seen in this framework. However, we focus on the revenue of the mechanism. Chen, Hassidim and Micali [3] consider the problem of maximizing revenue in multi-stage subgame perfect Nash implementation. Their goal is to design a robust mechanism that extracts a revenue of maximum social welfare. However, their mechanism is fairly complex, consisting of $n+1$ rounds, where n is the number of bidders.

The main ingredient we use to design our mechanism is the proportional sharing mechanism, which was introduced and studied by Kelly [9]. Most of the works in this line is about the social welfare of the systems (see the survey [7]). The revenue of proportional sharing mechanism is first considered in [14] and the revenue of a more general class mechanism called quasi proportional sharing is studied recently in [12, 15].

Structure of the paper. In Section 2 we give notations and some preliminary results. In Section 3 we prove a revenue upper bound for truthful mechanisms. Section 4 introduces our revenue benchmark and a competitive mechanism. Some proofs are given in the Appendix.

2 Notations and preliminaries

2.1 Notations

In this paper we consider auctions for n bidders in single parameter environments, i.e. bidder i's valuation for receiving service is v_i, and the valuation for not receiving service is normalized to be 0. Let $0 \leq x_i \leq 1$ represent the probability that the service is allocated to bidder i, and let p_i be his payment. We assume that bidders have quasi-linear utilities expressed $v_i x_i - p_i$.

We consider a setting called *general single parameter auction* [6], where the constraints of the service that can be allocated to bidders is represented by a set system representing feasible sets. A feasible set is a set of bidders that can be served simultaneously. We focus on the typical case of downward-closed environment where every subset of a feasible set is again feasible. For example, in single item auction the feasible set system contains singletons. Another example of such an environment is a combinatorial auction with single-minded bidders, where feasible sets correspond to subsets of bidders seeking disjoint bundles of goods.

If we consider randomized outcomes in this setting, then the feasible allocation vectors $\vec{x} = (x_1, ..x_n)$ form a polyhedron. We give the formal statement in the following theorem, whose proof can be found in Appendix A.1.

Theorem 2.1 *The set of randomized allocation vectors of the general single parameter auction is the set of non negative vectors $\vec{x} = (x_1, .., x_n)$ satisfying $A\vec{x} \leq \vec{1}, \vec{x} \geq \vec{0}$, where A is a non negative matrix.*

In the rest of the paper, we will use α_i^e as the entries of the matrix A, where $e \in E$ is a row of the matrix. The polyhedral environment can be written as

$$\sum_i \alpha_i^e x_i \leq 1 \text{ for all } e \in E, \text{ and } x_i \geq 0.$$

2.2 Truthful mechanisms in single parameter settings

The following result is the basic result in mechanism design.

Theorem 2.2 ([1, 13]) *A mechanism is truthful in expectation if and only if, for any bidder i and any fixed choice of bids by the other bidders b_{-i},*

(i) *$x_i(b_i)$ is monotone non-decreasing.*

(ii) *$p_i(b_i) = b_i x_i(b_i) - \int_0^{b_i} x_i(z)dz$.*

Given this theorem, the payment can be derived from the allocation rule. It is useful to specialize the theorem above to the case where the mechanism is deterministic, that is $x_i \in \{0, 1\}$. It is straightforward to see that deterministic truthful mechanisms are of the following types.

Corollary 2.1 *Any deterministic truthful auction is specified by a set of functions $t_i(b_i)$ which determine, for each bidder i and each set of bids b_i, an offer price to bidder i such that bidder i wins and pays price t_i if $b_i > t_i$, or loses and pays nothing if $b_i < t_i$. (Ties can be broken arbitrarily.)*

2.3 Digital good auctions

For digital goods auctions, the goods are in unlimited supply. The setting can be described as $0 \leq x_i \leq 1$ for all i. The following profit benchmark, call $\mathcal{F}_2$ is introduced in [4].

Definition 2.1 *The optimal single priced profit with at least two winners is*

$$\mathcal{F}_2(v) = max_{i \geq 2} i v_{(i)},$$

where $v_{(i)}$ is the ith largest valuation.

In the following we also use the notation $\mathcal{F}(v)$ defined as $\mathcal{F}(v) = max_i i v_{(i)}$.

Also in [4], Goldberg et al. designed the following mechanism and show that the revenue of this mechanism is at least 1/4 of $\mathcal{F}_2$.

Definition 2.2 (RSPE) *The Random Sampling Profit Extraction auction (RSPE) works as follows:*

(i) Randomly partition the bids b into two by flipping a fair coin for each bidder and assigning her to b' or b''. Compute $R' = \mathcal{F}(b')$ and $R'' = \mathcal{F}(b'')$, the optimal profits for each part.

(ii) Find the largest group of bidders in b'' that can share R' equally. Charge each bidder R'/k", where k" is the number of winning bidders.

Find the largest group of bidders in b' that can share R'' equally. Charge each bidder R''/k', where k' is the number of winning bidders.

Theorem 2.3 ([4]) *The RSPE generates a revenue at least 1/4 of $\mathcal{F}_2$.*

For completeness a proof of this theorem is given in the Appendix A.2.

2.4 Proportional sharing mechanism for polyhedral environments

Proportional sharing [9] is a natural mechanism, that can be generalized to polyhedral environments [8, 14]. When sharing a single resource with constraint $\sum_i \alpha_i x_i \leq 1$ the fair sharing mechanism requires that each bidder j submits a bid b_j, the amount of money she wants to pay, and the resource is allocated proportional to the bids, as $x_j = b_j/(\alpha_j \sum_i b_i)$.

For environments with more constraints $\sum_i \alpha_i^e x_i \leq 1$ for all $e \in E$, the mechanism requires that bidders submit bids b_j^e separately on each constraint e. We denote the sum of the bid $\sum_i b_i^e$ by p^e. The allocation rule limits the value x_j for bidder j to at most $x_j^e = b_j^e/(\alpha_j^e p^e)$. The idea is to ask bidders to submit bids b_j^e for each constraint e, allocate the resources separately, make bidder j pay $w_j = \sum_e b_j^e$, and then set $x_j = \min_{\{e:\alpha_j^e \neq 0\}} x_j^e$. The mechanism also needs to deal with constraints that are under-utilized by allowing each bidder to request an amount r_j^e without any monetary bid.

The mechanism can be described formally as follows:

Definition 2.3 (General Proportional Sharing [8, 14]) *Each bidder j submit a bid b_j^e and a request r_j^e for each constraint e. For constraint e we use the following allocation:*

- *If $\sum_i b_i^e > 0$ then $x_j^e = \frac{b_j^e}{\alpha_j^e(\sum_i b_i^e)}$ for $\forall j$.*
- *If $\sum_i b_i^e = 0$ and $\sum_i \alpha_i^e r_i^e \leq 1$ then $x_j^e = r_j^e$ for $\forall j$, else, set $x_j^e = 0$ for $\forall j$.*

For each bidder j, the amount of money that she needs to pay is $w_j = \sum_e b_j^e$ and the final allocated $x_j = \min_{\{e:\alpha_j^e \neq 0\}} x_j^e$.

It can be proved that a Nash equilibrium exists and give the following conditions. For completeness we provide a proof in Appendix A.3.

Theorem 2.4 ([8]) *There always exists a Nash equilibrium in the game defined by the General Proportional Sharing. Let $a_i = \max\{x_i | Ax \leq 1; x \geq 0\}$, $p^e = \sum_i b_i^e$. An allocation $\vec{x}$ a bid and a request vector $\vec{b}, \vec{r}$ is a Nash solution* **if and only if**:

$$v_j = \sum_e \frac{p^e \alpha_j^e}{(1 - \alpha_j^e x_j)} \quad for \quad 0 < x_j < a_j;$$

$$v_j \geq \sum_e \frac{p^e \alpha_j^e}{(1 - \alpha_j^e x_j)} \quad for \quad x_j = a_j \qquad and$$

$$v_j \leq \sum_e p^e \alpha_j^e \quad for \quad x_j = 0.$$

In the formula above we do not consider the constraint e that has only one positive coefficient α_i^e.

There is an intuitive way to understand the complex formula above by a "price taking" game, where $\{p^e = \sum_i b_i^e, e \in E\}$ are seen as unit prices and bidders optimize their payoff assuming that these prices are fixed. The condition above, however takes into account that p^e depends on the bid vector. For more detail, see the Appendix A.3.

3 Revenue upper bound of truthful mechanisms

In this section we describe an auction setting and prove an upper bound for the revenue of any truthful mechanism. The auction setting is the following example. The service provider can either provide service to a single bidder (numbered 0) or any subset of other bidders (numbered from 1 to n). This setting can be captured by the following inequality system

$$x_0 + x_i \leq 1 \ \ \forall\, i \in [1, .., n].$$

This inequality system captures exactly the following network bandwidth sharing game. Bidder 0 is interested in a path of bandwidth x_0 containing n different edges $e_1, .., e_n$, each with capacity of 1. Bidder $i, 1 \leq i \leq n$, is only interested in a path containing single edge e_i. See Figure 1.

Theorem 3.1 *In the auction setting described in Figure 1, for every truthful mechanism in expectation and any constant c, there exists a valuation vector $\vec{v}$ such that, $\sum_{i=1}^n v_i \geq \log n - 2$ and $\log\log n \leq v_0 \leq \log n$,* *and the revenue obtained by the mechanism is at most* $c(\frac{v_0}{\log v_0}) + 2$.

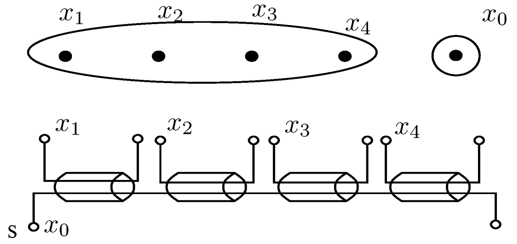

Figure 1: An example of general single parameter auction and bandwidth sharing game.

Proof. We first show that for every truthful mechanism, there exists a valuation $\log\log n \le v_0 \le \log n$ such that the revenue obtained from bidder 0 is at most $c(\frac{v_0}{\log v_0})$. Assume the contrary, $p_0(b) \ge c\frac{b}{\log b}$ for all $\log\log n \le b \le \log n$, we have

$$p_0(b) = b \cdot x_0(b) - \int_0^b x_0(t)\,dt.$$

Assume that x_0 is differentiable. Note that we can approximate the monotone function $x_0(b)$ by a differentiable function without essential change in the analysis of this proof. Taking the derivative of the formula above, we have

$$p_0'(b) = b \cdot x_0'(b) + x_0(b) - x_0(b) = b \cdot x_0'(b),$$

which means

$$x_0'(b) = \frac{p_0'(b)}{b}.$$

Now,

$$x_0(v) = \int_0^v x_0'(b)\,db = \int_0^v \frac{p_0'(b)}{b}\,db$$
$$= \frac{p_0(v)}{v} + \int_0^v \frac{p_0(b)}{b^2}\,db.$$

Thus,

$$x_0(v) \ge \int_0^v \frac{p_0(b)}{b^2}.$$

If $p_0(b) \ge c \cdot \frac{b}{\log b}$ for $\log\log n \le b \le \log n$,

$$\text{then } x_0(\log n) \ge c\int_{\log\log n}^{\log n} \frac{1}{b\log b}\,db =$$

$$= c(\log\log\log n - \log\log\log\log n).$$

This show that for every c, we can choose n large enough such that $x_0(\log n) > 1$, which is a contradiction.

We next show that there exists a valuation vector such that $\sum_{i=1}^n v_i \ge \log n - 2$, but the revenue obtained from these bidders is at most 2. If we know this, the theorem is proved. To show this we will use Yao's minimax principle [19]. This is a standard tool to reduce the analysis of randomized algorithms/ mechanisms to the analysis of deterministic ones on a distribution of input. We need to find a distribution on v_i, such that $\sum_i v_i \ge \log n - 2$ and the expected revenue of any *deterministic* truthful mechanism is at most 2.

Consider the random valuation $v_i = \frac{1}{k}$ with probability $\frac{1}{n}$ for $k = 1, .., n$. It is straightforward to see that any *deterministic* auction gains at most a revenue of $\frac{1}{n}$ revenue in expectation from a single bidder. Thus from n bidders, with independent valuations from this distribution, the expected revenue $E(R)$ is at most 1.

However, we need to show the revenue bound for a $\vec{v}$ such that $\sum_{i=1}^n v_i \ge \log n - 2$. One can use the Chebyshev's inequality for this purpose. We have

$$E(v_i) = \frac{1}{n}\sum_k \frac{1}{k} = \frac{H_n}{n}$$

and

$$\sigma^2(v_i) = \frac{1}{n}\sum_k (\frac{1}{k} - \frac{H_n}{n})^2 =$$

$$= \frac{1}{n}\sum_k \frac{1}{k^2} - (\frac{H_n}{n})^2 \le \frac{2}{n}.$$

We have

$$Pr(\sum_{i=1}^n v_i < \log n - t) < Pr(\sum_{i=1}^n v_i < H_n - t) \le$$

$$\le \frac{n\sigma^2}{t^2} \le \frac{2}{t^2}.$$

Thus we obtain

$$Pr(\sum_{i=1}^n v_i \ge \log n - 2) \ge \frac{1}{2}.$$

Now, we have

$$E(R) =$$

$$E(R|\sum v_i < \log n - 2) \cdot Pr(\sum v_i < \log n - 2)$$

$$+E(R|\sum v_i \ge \log n - 2) \cdot Pr(\sum v_i \ge \log n - 2).$$

Which implies

$$E(R|\sum v_i \geq \log n - 2) \leq$$
$$\leq \frac{E(R)}{Pr(\sum v_i \geq \log n - 2)} \leq \frac{1}{1/2} = 2.$$

By this we conclude the proof. □

Remark We will see in the next section that when the valuations v_i satisfy the condition in Theorem 3.1, the proportional sharing obtains at least $O(v_0)$ revenue. It is worth to describe this mechanism in this particular example. Consider the network bandwidth sharing interpretation of our example in Figure 1, bidder 0 bids a non negative vector: a number b_0^i on each edge of the graph and bidder $i, 1 \leq i \leq n$, only bids b_i on the edge e_i. The mechanism will use the fair sharing on each link. Bidder $i, 1 \leq i \leq n$ pays b_i and gets $x_i = \frac{b_i}{b_i + b_0^i}$. Bidder 0 pays $\sum_i b_0^i$ and gets $x_0 = \min_i \frac{b_0^i}{b_i + b_0^i}$.

4 Our mechanism

In this section we will introduce our revenue benchmark and give a competitive mechanism As discussed in the introduction, we observe that the general single parameter auction setting generalizes the cases of auctions for singe item and digital goods. Our mechanism can be thought of as the combination of the two special cases above. We first define the revenue benchmark. See Figure 2 for an illustration.

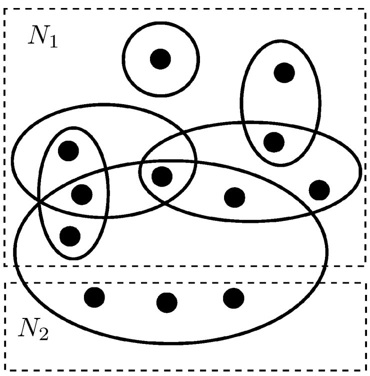

Figure 2: A revenue benchmark for downward closed set systems.

Definition 4.1 (Benchmark $\mathcal{R}$) $\mathcal{R} = \min_{N_1,N_2}$ *SocialOpt*$(N_1) + \mathcal{F}_2(N_2)$, *where* $N = N_1 \uplus N_2$ *is a partition the bidders, such that all bidders in* N_2 *can be serviced simultaneously.*

Remark This benchmark gives exactly the second valuation in the single item auction and the $\mathcal{F}_2$ benchmark in the digital good auction. In the setting of Theorem 3.1, one can show that $\mathcal{R} \geq \min\{v_0, \sum_{i=1}^n v_i\} \geq v_0 - 2$. Note that the revenue that we would like to obtain from N_2 is the maximal social welfare, therefore, in order to design a mechanism obtaining a constant factor of $\mathcal{R}$, we cannot fix the partition before running the mechanism. We will see later that the partition is part of the outcome of the mechanism.

In the rest of the paper, we give a mechanism whose revenue at Nash equilibrium is at least a constant factor of the Benchmark $\mathcal{R}$. This result combined with the revenue upper bound on truthful mechanisms shows that for the goal of maximizing revenue, Nash implementation can asymptotically generate more revenue than truthful mechanisms.

The ideas. The basic idea is to use the proportional sharing mechanism for general polyhedral environments as a version of creating competition among bidders, and then we give additional resources to bidders for extra money. More precisely, after the proportional sharing mechanism, we consider the bidders who get a large share of resources, which we call *big bidders*. Note that the set of big bidders depends on the proportional mechanism. The intuition is that there is a lack of competition among these bidders, and therefore, we can use the mechanism designed for the benchmark $\mathcal{F}_2$ for the big bidders. However, there are some issues with this approach. Because the second mechanism is run on the outcome of the first mechanism, bidders might behave differently from the case where the two mechanism are run separately.

The first problem is that, because the second phase of the mechanism is run only for the set of big bidders, therefore, the *small bidders* might overbid in the first round to get to the second one. To overcome this difficulty, we modify the paying scheme in the second round of the mechanism. The price that a bidder needs to pay is the maximum of the two values: the price obtained in the second round and a price related to the price that the bidder pays in the first round. By doing this we make sure that if the small bidders overbid in the first round, they still need to pay a large money in the second round, and their payoff will be negative if he does so. We define formally the modified version of RSPE (Definition 2.2) as follow.

Definition 4.2 (RSPE*$(\vec{p})$) *Given a price* p_i *for each bidder* i*. Let* b_i *be the bid from bidder* i*.*

(i) *Randomly partition the bids b into two by flipping a fair coin for each bidder and assigning her to b' or b''. Compute $R' = \mathcal{F}(b')$ and $R'' = \mathcal{F}(b'')$, the optimal profits for each part.*

(ii) *Find the largest group of bidders among b' that can equally share the profit (R''), the number of these bidders is k'. Charge bidder i $\max\{\frac{R''}{k'}, p_i\}$. Find the largest group of bidders among b'' that can equally share the profit (R'), the number of these bidders is k''. Charge bidder i $\max\{\frac{R'}{k''}, p_i\}$.*

The second problem is that in the first round the large bidders might also bid differently from the equilibrium of proportional sharing to change the set of bidders that survive to the second round, and thus the price of the second mechanism might be different and better for them. For this problem, we will modify the proportional sharing such that at an equilibrium, if a bidder gets a large share of the resource, then by bidding differently from the equilibrium, he cannot benefit in the second round of the mechanism.

To make it more precise, consider the simple case of sharing an unit of a single resource. We would like to modify the proportional sharing such that the following is true. Consider the set of bidders who get at least c fraction of the resource *at a Nash equilibrium*, if any of these bidders lowers his bid, he will get less than c, furthermore, he cannot change the set of the big bidders by over bidding. Note that this condition does not hold in traditional proportional sharing because by overbidding a bidder can change the price of the resource and other bidders will get less resource. To this end, we introduce the following mechanism called *Truncated Proportional Sharing* (TPS).

Definition 4.3 (TPS(c)) *The Truncated Proportional Sharing mechanism for the resource constraint $\sum_i \alpha_i x_i \leq 1$, and an upper limit c is the following.*

Each bidder i bids b_i. Let

$$b_i^* = \begin{cases} b_i & if \frac{b_i}{\alpha_i \cdot \sum_j b_j} \leq c, \\ b \text{ s.t } \frac{b}{\alpha_i \cdot (b + \sum_{j \neq i} b_j)} = c & if \frac{b_i}{\alpha_i \cdot \sum_j b_j} > c. \end{cases}$$

The payment for bidder i is b_i, and the allocation for bidder i is $x_i = \min\left\{c, \frac{b_i^*}{\alpha_i \cdot \sum_j b_j^*}\right\}$.

As the name of the mechanism suggests, the Truncated Proportional Sharing mechanism above is a modified version of traditional proportional sharing, where the resource is truncated by c, and the bid b_i is also truncated by a value at which bidder i gets c fraction of the resource. Thus, we can see that at a Nash equilibrium, no bidder i bids more than b_i^*, furthermore, if he bids less than b_i^*, then $x_i < c$ and he cannot change the set of big bidders by bidding more than b_i^*.

Now because at a Nash equilibrium, $b_i = b_i^*$, to analyze the Nash equilibria, we can see this game as a proportional sharing game discussed in Section 2. Observe that the resource that each bidder i can get is $\min\left\{c, \frac{b_i}{\alpha_i \cdot \sum_j b_j}\right\}$. This is exactly description of the game where each bidder has two constraints $x_i \leq c$ and $\sum_i \alpha_i x_i \leq 1$. Thus with the argument above and applying the basic result of Theorem 2.4, we obtain the following.

Lemma 4.1 *Assuming the valuation of bidder i is v_i, there is an unique Nash equilibrium of the mechanism TPS and the condition for the equilibria is $b_i = b_i^*$ for all i, furthermore, let $p = \sum_i b_i$, then the following is true.*

$$v_i = \frac{p\alpha_i}{(1 - \alpha_i x_i)} \; for \; 0 < x_i < c \; ;$$

$$v_i \leq \frac{p\alpha_i}{(1 - \alpha_i x_i)} \; for \; x_i = 0 \; ;$$

$$v_i \geq \frac{p\alpha_i}{(1 - \alpha_i x_i)} \; for \; x_i = c \; .$$

Moreover, if bidder i gets c fraction of the resource, then by increasing his bid, he does not influence other bidders' strategies and by lowering his bid, he gets less than c fraction of the resource. □

Remark The uniqueness of Nash equilibrium in the Lemma above can be seen directly from the condition of the equilibrium. The $\vec{x}$ satisfying this condition is the optimal point of the convex optimization program $\max \sum_i (v_i x_i - \frac{1}{2}\alpha_i v_i x_i^2)$ subject to $\sum_i \alpha_i x_i \leq 1$ and $0 \leq x_i \leq c$. This mechanism for a single constraint and this lemma is the building block of the mechanism for a more complex polyhedron.

We are now ready to define our main mechanism, called Two-Phase Mechanism.

Definition 4.4 (TPM(c_1, c_2)) *The Two-Phase Mechanism is for a general downward closed set system, which is captured by the polyhedral $A\vec{x} \leq \vec{1}$, each constraint (row) e of A is $\sum \alpha_i^e x_i \leq 1$. The mechanism uses the parameters c_1, c_2, where $\frac{c_1}{2} < c_2 < c_1 < 1$. These parameters will be chosen later to optimize the revenue bound. The mechanism consists of two phases:*

(i) *Run proportional sharing for the environment* $\frac{1}{c_1} \cdot A\vec{x} \leq \vec{1}$*, but use the Truncated Proportional Sharing TPS(*c_2*) on each constraint. At the end of the game(Nash equilibrium), we obtain an allocation vector and a price* p^e *on every constraint* e *and the bid vector* $\vec{b}$ *at Nash equilibrium.*

(ii) For the remaining resource $\frac{1}{1-c_1}A\vec{x} \leq \vec{1}$*, let* $p_i = \frac{1-c_1}{c_1-c_2} \cdot \sum_e \alpha_i^e p^e$*. On the bidders that obtained* c_2 *in the first round, run RSPE*(*$\vec{p}$*), where the winning bidders get* $1 - c_1$ *(instead of 1 as in Definition 4.2).*

Our main result is the following.

Theorem 4.1 *Given an arbitrarily small* ϵ*, there are proper parameters* c_1, c_2 *such that the revenue at Nash equilibrium of the mechanism TPM with these parameter is at least* $\frac{\mathcal{R}}{14+\epsilon}$*, where* $\mathcal{R}$ *is the benchmark defined in Definition 4.1.*

Remark The second phase of our mechanism is truthful. Below, we will see that the equilibria in the first round needs to satisfy some conditions. Therefore, if in the *truthful* mechanism of the second round bidder bid truthfully, then all equilibria resulted by our mechanism generates at least a constant factor of the benchmark $\mathcal{R}$.

We first derive a condition for a Nash equilibrium of the general polyhedral settings. As described above, we will consider the first phase of the mechanism as if there were no second round. We then claim that this condition also holds for Nash equilibrium of the extended game with the second round. The precise statement is the following.

Lemma 4.2 *Consider the mechanism TPM in Definition 4.4. An allocation* $\vec{x}$ *and a bid vector* $\vec{b}$ *of the first round is in a Nash solution* **if and only if**

$$\sum_i \alpha_i^e x_i \leq c_1 \quad \textit{and} \quad 0 \leq x_i \leq c_2 \ \ \forall i \quad \textit{and} \quad e \in E;$$

$$v_j \geq \sum_e p^e \frac{\alpha_j^e/c_1}{(1-\alpha_j^e x_j/c_1)} \quad \textit{for} \quad x_j = c_2;$$

$$v_j \leq \sum_e p^e \frac{\alpha_j^e}{c_1} \quad \textit{for} \quad x_j = 0;$$

$$v_j = \sum_e p^e \frac{\alpha_j^e/c_1}{(1-\alpha_j^e x_j/c_1)} \quad \textit{for} \quad 0 < x_j < c_2$$

where, $p^e = \sum_i b_i^e$*.*

Proof. To see that this is the condition of a Nash equilibrium without the second phase of the mechanism, one can give a proof, which is exactly the same as the proof of Theorem 2.4. We note here that $A\vec{x} \leq \vec{1}$ is the polyhedral form of a downward closed set system, and we assume each bidder alone can receive the service (otherwise, we can ignore this bider). Therefore, $\max\{x_i : A\vec{x} \leq \vec{1}\} = 1$, hence in the environment $\sum_i \alpha_i^e x_i \leq c_1$ and $0 \leq x_i \leq c_2$, the maximum resource that each bidder can get is c_2.

We now need to see that with the second round, this condition is still the condition for Nash equilibrium of the extended mechanism. Because of Lemma 4.1, if a big bidder decreases his bid, he will get less than c_2 and will not be able to enter the second round and he would not increase the bid either, because by doing so, he will need to pay more, but cannot affect the strategies of any other bidders.

For the small bidders, we will show that if he increases his bid in Nash equilibrium to get c_2 unit of resource to enter the second round the unit price that he needs to pay is larger than his valuation. Because we know that $x_i = 1$ is a feasible solution, thus $\alpha_i^e \leq 1$ for all i, e. Observe that if $0 \leq x_i < c_2$, then

$$v_i = p^e \sum_e \frac{\alpha_j^e/c_1}{(1-\alpha_j^e x_j/c_1)} \leq$$

$$\leq p^e \sum_e \frac{\alpha_j^e/c_1}{(1-c_2/c_1)} = \frac{1}{c_1-c_2} \sum_e p^e \alpha_i^e.$$

Now, if bidder i increases the bids, p^e will also increase, and because in the second round, the price per unit is at least $\frac{1}{c_1-c_2} \sum_e p^e \alpha_i^e$ for bidder i. Thus, bidder i cannot benefit from overbidding. □

We now show a lemma which bounds the revenue obtained in the first round with the optimal social welfare of the smaller bidders.

Lemma 4.3 *Let* N_0 *be the set of bidders whose valuation* $v_i \leq \frac{1}{c_1-c_2} \sum_e p^e \alpha_i^e$*. Let* N_1 *be the set of bidders obtaining less than* c_2*, then we have* $N_1 \subset N_0$ *and*

$$\sum_e \frac{1}{c_1-c_2} p^e \geq \max_{z \geq 0: Az \leq 1} \sum_{i \in N_0} v_i z_i \geq$$

$$\geq \max_{z \geq 0: Az \leq 1} \sum_{i \in N_1} v_i z_i.$$

Proof. Similar to the prove above, for $0 \leq x_i < c_2$, we have

$$v_i = p^e \sum_e \frac{\alpha_j^e/c_1}{(1-\alpha_j^e x_j/c_1)} \leq$$

$$\leq p^e \sum_e \frac{\alpha_j^e/c_1}{(1-c_2/c_1)} = \frac{1}{c_1-c_2}\sum_e p^e \alpha_i^e.$$

This shows that $N_1 \subset N_0$. To show the inequality, we use the duality theorem

$$\max_{z \geq 0}\left\{\sum_i v_i z_i : Az \leq 1\right\} \leq$$

$$\leq \min_{w \geq 0}\left\{\sum_e w^e : \sum_e w^e \alpha_i^e \geq v_i\right\}.$$

Recall that α_i^e are the entries of matrix A. Applying this duality lemma, in our case $w^e = \frac{p^e}{c1-c_2}$. Thus, we have

$$\sum_e \frac{1}{c_1-c_2} p^e \geq \max_{z:Az\leq 1} \sum_{i \in N_0} v_i z_i.$$

□

We are now ready to prove our main theorem.

Proof of Theorem 4.1 The mechanism in the first round gives us a partition of the bidders into N_1 and N_2, where N_2 is the set of big bidders, who get $c_2 > c_1/2$ fraction of the resource, and N_1 is the set of the remaining bidders (small bidders). Let $\vec{y}$ be an allocation vector of the first round and $\vec{z}$ of the second round of the mechanism. Let R_1, R_2 be the expected revenue obtained in the first and second round, relatively.

We first show that the large bidders form a feasible set, that is, they can be served simultaneously. Recall that an allocation needs to satisfy $x_i + x_j \leq 1$, whenever bidder i and j do not belong to a feasible set. In the first round of the mechanism, we scaled the resource down by c_1, therefore, if an allocation vector $\vec{y}$ satisfies $y_i + y_j > c_1$, then i, j belong to a feasible set and can be served simultaneously. We choose the set N_2 to be the bidders who get $c_2 > c_1/2$, therefore, N_2 is a feasible set.

Next, we show that the final allocation vector $(\vec{y}+\vec{z})$ is feasible, that is $A(\vec{y}+\vec{z}) \leq \vec{1}$. In the first round we have that $A\vec{y} \leq \vec{c_1}$. The second round allocates resource to the bidders in N_2. As shown above that N_2 is a feasible set. This means that the allocation vector $\vec{1}_{N_2}$, which corresponds to servicing all the bidders in N_2, satisfies $A\vec{1}_{N_2} \leq \vec{1}$. However, in the second round of our mechanism, we allocate to each bidder at most $1 - c_1$, therefore, $A\vec{z} \leq (1-c_1)\vec{1}$. From this we have $A(\vec{y}+\vec{z}) \leq \vec{1}$, which we need to show.

Finally, we prove an lower bound on the revenue of our mechanism. According to Lemma 4.3, the revenue obtained in the first round is at least

$$R_1 \geq (c_1 - c_2) \max_{z:Az\leq 1} \sum_{i \in N_1} v_i z_i.$$

Therefore,

$$\frac{R_1}{c_1 - c_2} \geq \text{SocialOpt}(N_1). \tag{1}$$

In the second round of the mechanism we run a mechanism to extract $\mathcal{F}_2(N_2)$ from the bidders in N_2 (scaled by $1 - c_1$). Using Theorem 2.3, one would expect to have $\frac{R_2}{1-c_1} \geq \frac{\mathcal{F}_2(N_2)}{4}$. However, the mechanism we use in the second round is slightly different from the mechanism RSPE of Definition 2.2. The bidder i's payment for $1 - c_1$ of the resource is the maximum of the price he would need to pay in the original RSPE mechanism scaled by $1-c_1$ and $\frac{1-c_1}{c_1-c_2} \cdot \sum_e \alpha_i^e p^e$. Therefore, we would get 0 revenue from bidder i with $v_i < \frac{1}{c_1-c_2} \cdot \sum_e \alpha_i^e p^e$. However, according to Lemma 4.3, we have that $\frac{R_1}{c_1-c_2}$ is at least the optimal social welfare of these bidders, hence if we would have a weaker inequality as follow

$$\frac{R_1}{c_1-c_2} + \frac{R_2}{1-c_1} \geq \frac{\mathcal{F}_2(N_2)}{4}. \tag{2}$$

Thus, combining (1) and (2), we have

$$\frac{5}{c_1-c_2}R_1 + \frac{4}{1-c_1}R_2 \geq \text{SocialOpt}(N_1) + \mathcal{F}_2(N_2).$$

Choosing $c_1 = 5/7, c_2 = 5/14 + \epsilon'$, where ϵ' is positive but negligible, one have

$$\frac{5}{5/14-\epsilon'}R_1 + \frac{4}{2/7}R_2 \geq \text{SocialOpt}(N_1) + \mathcal{F}_2(N_2).$$

Thus for any $\epsilon > 0$, we can choose $\epsilon' > 0$ such that

$$(14+\epsilon)(R_1+R_2) \geq \text{SocialOpt}(N_1) + \mathcal{F}_2(N_2).$$

This is what we need to prove. □

5 Conclusions

We have introduced a new mechanism: a combination of two natural and well known classes of auctions. It is proved that our mechanism is completive against a worst-case revenue benchmark that no truthful auction can obtain a constant approximation.

We believe this is only a beginning step in the direction of designing non-truthful mechanisms. There are several open questions, such as, defining a more systematic framework for natural mechanisms in full information Nash implementation settings; defining better benchmarks and designing simple mechanisms that bidders can use decentralized learning dynamics to find Nash equilibria.

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A Appendix

A.1 Proof of Theorem 2.1

Denote a $\{0,1\}$ vector that corresponds to the allocation that gives service to all bidder in a feasible set S by $\vec{1}_S$. A randomized outcome that output $\vec{1}_S$ with probability p_S is $\sum_S p_S \vec{1}_S$. Thus, all possible randomized outcome is the set of the convex hull the vectors $\vec{1}_S$, for all feasible set S. We call this convex hull $\mathcal{H}$.

We need to show that if the set system is downward-closed then there is a nonnegative matrix A such that the convex hull can be captured as $\mathcal{H} = \{\vec{x} : A\vec{x} \leq \vec{1}, \vec{x} \geq \vec{0}\}$.

In order to see this, we need to show that, given a vector $\vec{x} \in \mathcal{H}$ then any vector $\vec{0} \leq \vec{y} \leq \vec{x}$ is also in the convex hull. If we know this fact then by simple facts in convex geometry, one can see that that for every $z \notin \mathcal{H}$, there exists a *non-negative* vector $\vec{a}$ such that $\vec{a}^T\vec{w} \leq 1 < \vec{a}^T\vec{z}\ \forall \vec{w} \in \mathcal{H}$. And this will prove our theorem.

Now, for $\vec{x} \in \mathcal{H}$ and $\vec{0} \leq \vec{y} \leq \vec{x}$, we need to show that $\vec{y} \in \mathcal{H}$. We will prove this by induction on the number of coordinate of $\vec{y}$ that are smaller than the corresponding coordinates of $\vec{x}$. Let i be a coordinate such that $y_i < x_i$. Consider the vector $\vec{x'}$ and $\vec{x''}$ whose all the coordinates are equal to $\vec{x}$, except $x'_i = y_i$ and $x''_i = 0$ respectively. First, $\vec{x'}$ is in the convex hull of $\vec{x}$ and $\vec{x''}$. Secondly, if $\vec{x} = \sum p_S \vec{1}_S$ then $\vec{x''} = \sum p_S \vec{1}_{S-i}$ and because the property of the downward closed system, we have $\vec{x''} \in \mathcal{H}$. Therefore $\vec{x'} \in \mathcal{H}$.

Next we use the induction step. $\vec{y} \leq \vec{x'}$, $\vec{x'} \in \mathcal{H}$ and the number of coordinate j of $\vec{y}$ such that $y_j < x'_j$ is strictly less than when compared with $\vec{x}$. By this we finished the proof.

A.2 Proof of Theorem 2.3

As we discussed above, the profit of RSPE is min(R',R"). Thus, we just need to analyze $E[\min(R', R'')]$. Assume that $\mathcal{F}_2(b) = kp$ has with $k \geq 2$ winners at price p. Of the k winners in $\mathcal{F}_2$, let k' be the number of them that are in b' and k'' the number that are in b''. Because there are k' bidders in b' at price p, $R' \geq k'p$. Likewise, $R'' \geq k''p$. Thus,

$$\frac{E[RSPE(b)]}{\mathcal{F}_2(b)} = \frac{E[min(R', R'')]}{kp} \geq$$

$$\geq \frac{E[min(k'p, k''p)]}{kp} = \frac{E[min(k', k'')]}{k} \geq \frac{1}{4}.$$

The last inequality follows from the fact that if $k \geq 2$ fair coins (corresponding to placing the winning bidders into either b' or b'') are flipped then

$$E[\min\{\#heads, \#tails\}] = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} i \cdot \binom{k}{i} \frac{1}{2^k} \geq \frac{k}{4}.$$

The equality occurs when $k = 2$.

A.3 Proof of Theorem 2.4

We will analyze the condition for an equilibrium for this game and we will use these conditions to show that an equilibrium always exists. First, we will discuss the game with price taking strategies to gain some intuitions.

Price taking strategy. Kelly [9] has considered a version of this "game" when prices are assigned by the auctioneer, and bidders are "price takers" in the sense that they act to optimize their value at the given prices. We can also view our fair-sharing game as a pricing game, but in our game the prices are determined as part of the game. However, it is useful to compare the mechanism above with a game where bidders behave as price takers.

Consider an equilibrium of the game, it must be the case that $x^e_j = x_j$ for all constraints e that costs money, or otherwise bidder j can reduce her bid b^e_j without affecting her allocation. One way to think about the mechanism above is the following: Bidders decide on each constraint a price $p^e = \sum_j b^e_j$; now bidders have to pay for each constraint e at its unit price p^e. For each bidder i, when getting a share of x_i, he uses up $\alpha^e_i x_i$ on the constraint e, and hence needs to pay $p^e \alpha^e_i x_i$ on e. In order to get all the needed resources bidder i must pay a unit price of $\sum_e \alpha^e_i p^e$ for his resource.

Now, if we assume that the price p^e are given, then for each bidder i the unit price is fixed. Therefore to maximize her utility, bidder i will maximize his utility, that is $U_i(x_i) - \sum_e p^e \alpha^e_i x_i$. Taking the derivative in x_i to determine the optimal value for bidder i we see that bidder i will choose to buy an x_i such that: the derivative $U'_i(x_i)$ is equal to the unit price or in the case $U'_i(0)$ is less than the unit price, she will choose not to buy any resource. We rewrite this as follow:

$$U'_i(x_i) = \sum_e \alpha^e_i p^e \text{ OR}$$

$$x_i = 0 \quad \text{if } U'_i(0) < \sum_e \alpha^e_i p^e.$$

Nash condition. The issue by using the price taking game as an approximation for our mechanism is that the prices p^e are not fixed. For example, when a constraint e has only a single bidder having a positive coefficient α^e_i, then $p^e = 0$. For this types of constraints, we will consider them as upper bounds on maximum resource that bidder i can get (this will be discussed in more details below). We now consider the constraints e that have at least two positive coefficient α^e_i.

Consider a set of bids b^e_i, and a resulting allocation x, where bidder i gets allocation x_i. When is this allocation at equilibrium? For each constraint e we use $p^e = \sum_i b_i^e$, the sum of the bids, as the unit price of the constraint. We prove a more general version of the theorem, where each bidder has a concave, monotone increasing utility function $U_i(x_i)$.

Now consider the optimization problem of a bidder j assuming bids b_i^e for all other bidders are set. The bidder j is interested in maximizing her utility at $U_j(x_j) - \sum_e b_j^e$. At equilibrium, it must be the case that $x_j^e = x_j$ for all constraints e that costs money, or otherwise bidder j can reduce her bid b_j^e without affecting her allocation. So we can think of the bidder's optimization problem as dependent on one variable x_j, the allocation she will receive. What bid does bidder j have to submit for a constraint e to get allocation $x_j^e = x_j$? Bids must satisfy the following condition:

$$\text{If } b_j^e > 0 \text{ then: } \alpha_j^e x_j = \frac{b_j^e}{\sum_i b_i^e}.$$

Assuming all other bids b_i^e are fixed, we can express the bid b_j^e needed as follows.

$$b_j^e(x_j) = \frac{\alpha_j^e x_j \sum_{i\neq j} b_i^e}{1-\alpha_j^e x_j}.$$

Note that this expression assumes that $\alpha_j x_j < 1$, that is, j is not the only bidder on the constraint at equilibrium. It is not hard to see that this is guaranteed by having at least bidders for each constraint.

Bidder j will want to choose x_j to maximize her utility. For this end, it will useful to express the derivative of the bid b_j^e when viewed as a function of x_j. We get the following (again assuming $\alpha_j x_j < 1$):

$$\frac{\partial}{\partial x_j} b_j^e(x_j) = \frac{\alpha_j^e \sum_{i\neq j} b_i^e}{(1-\alpha_j^e x_j)^2}.$$

Substituting $\sum_{i\neq j} b_i^e = p^e(1-\alpha_j^e x_j)$ and simplifying we get that

$$\frac{\partial}{\partial x_j} b_j^e(x_j) = \frac{p^e \alpha_j^e}{1-\alpha_j^e x_j}.$$

Note that in the calculation above we assume that for all constraints e there are at least 2 bidders. For a constraint e that has only one bidder i (with positive α_i^e), then bidder i would bid 0 and request the maximum resource that he can get on this constraint, which is $\frac{1}{\alpha_i^e}$. Recall that a_i is the maximum resource that bidder i can get thus, $a_i \leq \frac{1}{\alpha_i^e}$. Therefore we can ignore these types of constraints and consider the following optimization problem of bidder j. She wants to maximize her utility $U_j(x_j) - \sum_e b_j^e$ subject to $x_j \leq a_j$, which can now be expressed as

$$\max_{x_j} U_j(x_j) - \sum_e \frac{\alpha_j^e x_j \sum_{j\neq i} b_i^e}{1-\alpha_j^e x_j}, \text{ sbjt. } x_j \leq a_j.$$

The function above is a concave function of x_j. Thus, there are three possibilities:

(i) the maximum occurs at a value x_j, where the derivative of this function 0,

(ii) if the derivative is negative everywhere, maximum occurs at $x_j = 0$,

(iii) $x_j = a_j$ if the derivative at a_j is at least 0. Using the derivatives we computed above, we get the derivative of bidder jth utility as a function of her allocation x_j to be

$$U_j'(x_j) - \sum_e \frac{p^e \alpha_j^e}{(1-\alpha_j^e x_j)}.$$

This derivative is a strictly decreasing function, so we have the following Nash condition

$$\begin{gathered}\textstyle\sum_i \alpha^e x_i \leq 1;\;\; x_i \geq 0 \text{ for all } e \in E \text{ and all } i;\\ U_j'(x_j) = \textstyle\sum_e \frac{p^e \alpha_j^e}{(1-\alpha_j^e x_j)} \text{ if } x_j > 0;\\ U_j'(0) \leq \textstyle\sum_e p^e \alpha_j^e \text{ if } x_j = 0;\\ U_j'(a_i) \geq \textstyle\sum_e \frac{p^e \alpha_j^e}{(1-\alpha_j^e x_j)} \text{ if } x_j = a_i.\end{gathered}$$

And in this condition, we ignore all the constraints e that has a single positive coefficient α_i^e.

To see that there is always a Nash equilibrium, observe the game we define above is a concave n-person game: each payoff function is continuous in the composite strategy vector $\vec{b_i}$, and the strategy space of each bidder is a compact, convex, nonempty subset of $\mathbb{R}^{|E|}$. Applying Rosen's existence theorem [18] (proved using Kakutani's fixed point theorem), we conclude that a Nash equilibrium exists for this game. By this, we finished the proof.