Optimal Testing of Reed-Muller Codes

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Reed-Muller Codes

- RM(n,d) = { f: $F_2^n \to F_2^n$
  f is a degree d polynomial }

- Dimension $\approx n^d$
- Distance = 2^{-d}

Testing Reed-Muller Codes

- An RM-test T
  given f: $F_2^n \to F_2$
  - If f $\in$ RM(n,d) then
    Pr[T(f) accepts] = 1
  - If Δ(f, RM(n,d)) > 0.1, then
    Pr[T(f) accepts] < 0.9

The subspace test

- [AKKLR03] proposed, analyzed T_{d+1}
  - Pick a random d+1 dimensional subspace A
  - Check that $f|_A$ is degree d

- Theorem: If $\Delta(f, RM(n,d)) > 0.1$ then

$$\Pr[T_{d+1}(f) \text{ rejects}] > O(1/2^d)$$

- Corollary: RM(n,d) is testable with $O(4^d)$ queries

The Question

- How many queries are needed to test RM(n,d)?
- $\geq 2^{d+1}$
  - no dual constraints on fewer coordinates
- $\leq O(4^d)$
  - by AKKLR

Main Result

- Theorem T_{d+1} is a tester for RM(n,d).

  Specifically, if Δ(f, RM(n,d)) > 0.1, then

  Pr[T_{d+1}(f) rejects] > 0.01

So O(2^d) queries suffice for testing RM(n,d).

T_{d+1}

- T_{d+1} doesn't do more
  - For infinitely many d,
  - $\exists$ functions f: $F_2^n \rightarrow F_2$ s.t.
    - Pr[T_{d+1}(f) rejects] = 0.3
    - Δ(f, RM(n,d)) = 1/2 - $o_n(1)$
  - Builds on "counterexamples to inverse Gowers conjecture" [LMS, GT]

Proof Plan

Induction on n

Structure of induction on n

Want to show:

- $\Delta(f, RM(n, d)) > 0.1$ $\Rightarrow$ Pr[T(f) rejects] > 0.01

By induction, for each hyperplane A:

- $\Delta(f|_A, RM(n-1,d)) > 0.1$ $\Rightarrow$ $\Pr[T(f|_A)$ rejects] > 0.01

Any Relation?

$\mathbf{E}_A[\Pr[T(f|_A) \text{ rejects}]] = \Pr[T(f) \text{ rejects}]$

Main Lemma

- Lemma Let $A_1, \ldots, A_K$ be hyperplanes with:
  - $K > 4 \cdot 2^d$
  - For each i,

$$\Delta(f|_{A_i}, RM(n-1, d)) < \beta < \tfrac{1}{4}\, 2^{-d}$$

Then,

$$\Delta(f, RM(n,d)) < 3\,\beta + (9/K)$$

Analysis of the RM test

- Induction claim:

  Δ (f, RM(n,d)) > (0.01) 2^{-d}

$\Rightarrow$ Pr[T(f) rejects] > 0.1 + $2^d 2^{-n}$

- Proof:
- Case 1 : If less than $4 \cdot 2^d$ hyperplanes A with

  $\Delta(f|_A$, RM(n-1,d)) < (0.01) 2^{-d}

- Case 2: Else, by the lemma:

  Δ(f, RM(n,d)) < 0.03 2^{-d} + 0.25 2^{-d}

f is moderately close to RM(n,d)

Actually …

- Induction analyzing the test T_{d+20}

- Claim:

$$\Pr[T_{d+20}(f) \text{ rejects}] \geq 2^{-20} \Pr[T_{d+1}(f) \text{ rejects}]$$

The Lemma

- We have hyperplanes $A_1, \ldots, A_K$, and polynomials $P_1, \ldots, P_K$ with

$$\Delta(f|_{A_i}, P_i) < \beta$$

- How to produce P on F_2^n?
  - The unique P consistent with all the P_i
  - (if any)

Getting P consistent with the P_i

- P_i and P_h are mutually consistent

  $$\Delta(P_i|_{A_i \cap A_h}, P_h|_{A_i \cap A_h}) < 2^{-d}$$

- Let $A_1, \ldots, A_{d+1}$ be *independent*
  - Make them $x_1 = 0, x_2 = 0, \ldots x_{d+1} = 0$
- Let

  $$P_i(x_{[1, d+1] - i}, y) = \sum_{S,\, i \notin S} c_{S,i}(y) \prod_{i \in S} x_i$$

- Mutual consistency $\Rightarrow c_{S,i} = c_{S,j}$ $(= c_S)$

  $$P(x_{[1, d+1]}, y) = \sum_{S} c_S(y) \prod_{i \in S} x_i$$

The Lemma III

- P consistent with the P_i on $A_1, \ldots, A_{d+1}$
- P consistent with the P_i on all A_i
- Claim: Δ(P, f) < 3 β + 9/K
  - $\geq$ (1 - 9/K) fraction of $x \in F_2^n$ are in > 1/3 A_j's
  - For such x, f(x) $\neq$ P(x) causes f(x) $\neq$ P_i(x) for $\geq$ 1/3 of the i's.