Hierarchy Theorems for Property Testing

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Meta-question to answer

M := model for combinatorial property testing

- Binary strings
- Bounded degree graphs/incidence lists
- Dense graphs/adjacency matrix

n := input size parameter

Q. : Possible testing complexities in *M*?

Question to answer

M = model for property testing

$q = q(n)$ – target testing complexity

$q(n) \leq$ max.complexity(M,n)

- n in binary strings
- dn in d-bounded degree graphs
- n^2 in dense graphs

Q. : Whether there exists a property P with testing complexity $\Theta(q)$?

The answer

YES!

- Testing complexity spectrum is continuous in each of the models

Binary strings

Th. 1: For every $q: \mathbb{N} \to \mathbb{N}$, $q(n) = O(n)$

there is a property Π of binary strings s.t.:

1. Π is testable in $q+O(1)$ queries
2. Π is not testable in $o(q)$ queries

Proof: simple

Π' := $\bigcup_{q \in \mathbb{N}} \Pi'_q$ - hard to test property [GGR]

Π: Π_n = repeating $f \in \Pi'_q$ n/q times

Bounded degree graphs

Th. 2: For every q: $\mathbb{N} \to \mathbb{N}$, $q(n) = O(n)$

there is a graph property Π s.t.:

1. Π is testable in $O(q)$ queries
2. Π is not testable in $o(q)$ queries

Proof: simple

Π' := 3-colorability – testing complexity $\Theta_d(|V(G|)$ in

d-bounded degree graphs [BOT]

Π: Π_n := all graphs on n vertices with max. degree d,

all connected components ≤ $q(n)$ and 3-colorable

Adjacency matrix model

Th. 3: For every q: $\mathbb{N} \rightarrow \mathbb{N}$, $q(n) = O(n^2)$

there is a graph property Π s.t.:

1. Π is testable in $O(q)$ queries
2. Π is not testable in $o(q)$ queries

(can make $\Pi \in \mathcal{P}$, tester is efficient)

Proof: not so simple technically

Main idea: blowing up a hard to test graph property

Adjacency matrix model -argument

Π' = $\bigcup_{q\in\mathbb{N}} \Pi'_q$ – graph property

1. in $\mathcal{P}$
2. testing complexity $\Theta(q^2)$
3. has an efficient tester

(can be derived from hard to test binary strings/linear codes with large dual distance)

Step 1: dispersing

Make sure: $u \neq v \in V(G) \Rightarrow |N(u) \Delta N(v)| = \Theta(|V(G)|$

Step 2: blowing up

$v \in V(G) \rightarrow$ cluster C_v, $|C_v| = n/\sqrt{q}$

$(u,v) \in E(G) \rightarrow$ complete bipartite graph between C_u and C_v

Adjacency matrix model –argument (cont.)

Important: blow-up operation does not preserve relative distance exactly (Matsliah; Pikhurko)

- But preserves it up to a constant factor (GKNR; Pikhurko)

Lower bound: Can reduce testing Π' to testing Π

- as rel. distance from Π' is essent. preserved when blowing up

Upper bound: 1. Guess basis of blow-up G'

(here use disperseness of G')

2. Test whether G is a blow-up of G'

(two-sided error)

3. Test whether $G' \in \Pi'$