Invariance in Property Testing

Madhu Sudan
Microsoft/MIT

Property Testing

- … of functions from D to R:
  - Property P µ {D → R}
- Distance
  - δ(f,g) = Pr$_{x\ 2\ D}$ [f(x) ≠ g(x)]
  - δ(f,P) = min$_{g\ 2\ P}$ [δ(f,g)]
  - f is ε-close to g (f ¼$_2$ g) iff δ(f,g) • ε.
- Local testability:
  - P is (k, ε, δ)-locally testable if 9 k-query test T
    - f 2 P) T^f accepts w.p. 1-ε.
    - δ(f,P) > δ) T^f accepts w.p. ε.
- Notes: want k(ε, δ) = O(1) for ε,δ= Ω(1).

Brief History

- [Blum,Luby,Rubinfeld – S'90]
  - Linearity + application to program testing
- [Babai,Fortnow,Lund – F'90]
  - Multilinearity + application to PCPs (MIP).
- [Rubinfeld+S.]
  - Low-degree testing
- [Goldreich,Goldwasser,Ron]
  - Graph property testing
- Since then … many developments
  - Graph properties
  - Statistical properties
  - …
  - More algebraic properties

Specific Directions in Algebraic P.T.

- More Properties
  - Low-degree ($d < q$) functions [RS]
  - Moderate-degree ($q < d < n$) functions
    - $q=2$: [AKKLR]
    - General q: [KR, JPRZ]
  - Long code/Dictator/Junta testing [BGS,PRS]
  - BCH codes (Trace of low-deg. poly.) [KL]

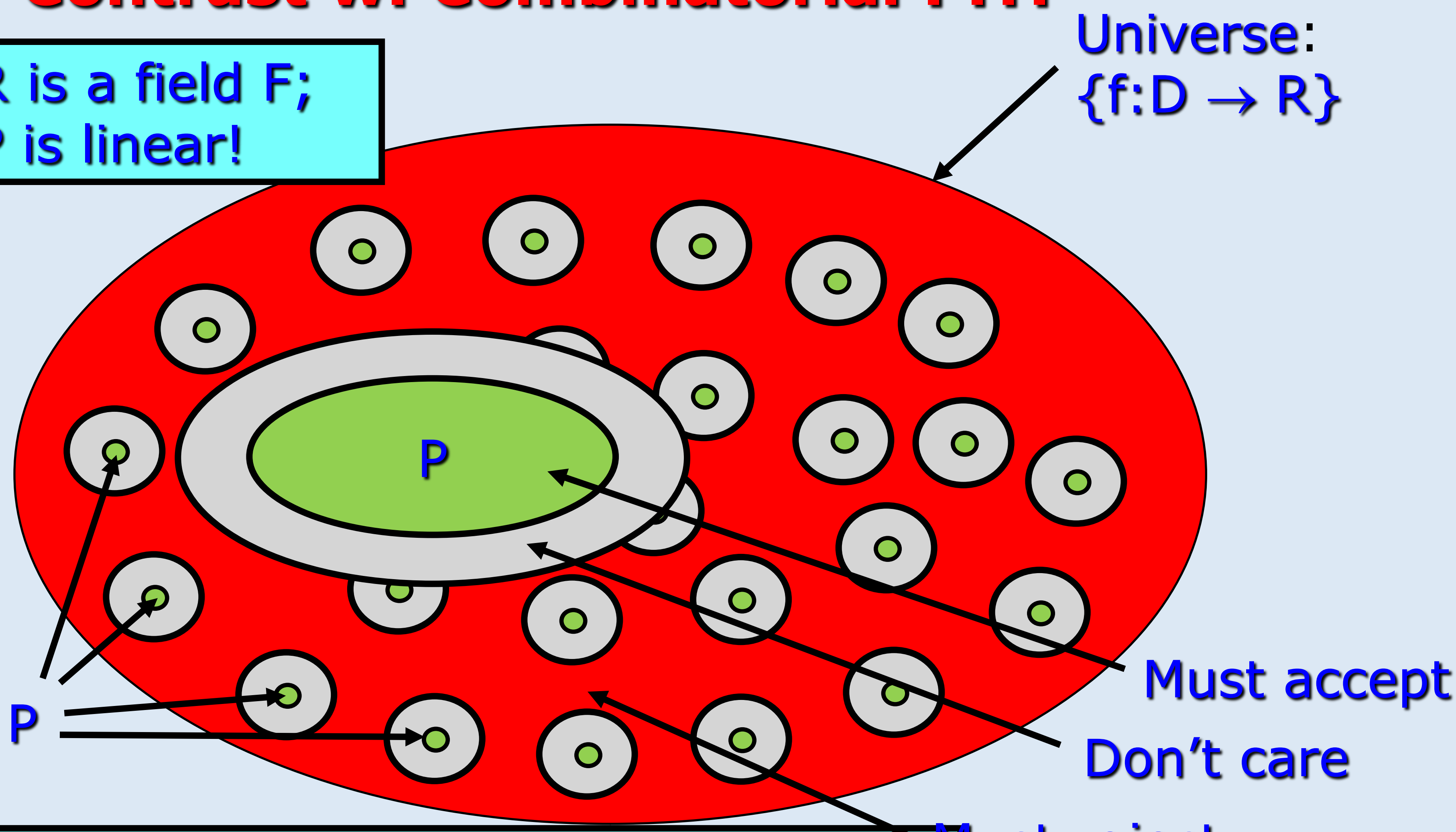
- Better Parameters (motivated by PCPs).
  - #queries, high-error, amortized query complexity, reduced randomness.

My concerns …

- Relatively few results …
  - Why can't we get "rich" class of properties that are all testable?
  - Why are proofs so specific to property being tested?
- What made Graph Property Testing so well-understood?
- What is "novel" about Property Testing, when compared to "polling"?

Contrast w. Combinatorial P.T.

R is a field F;
P is linear!



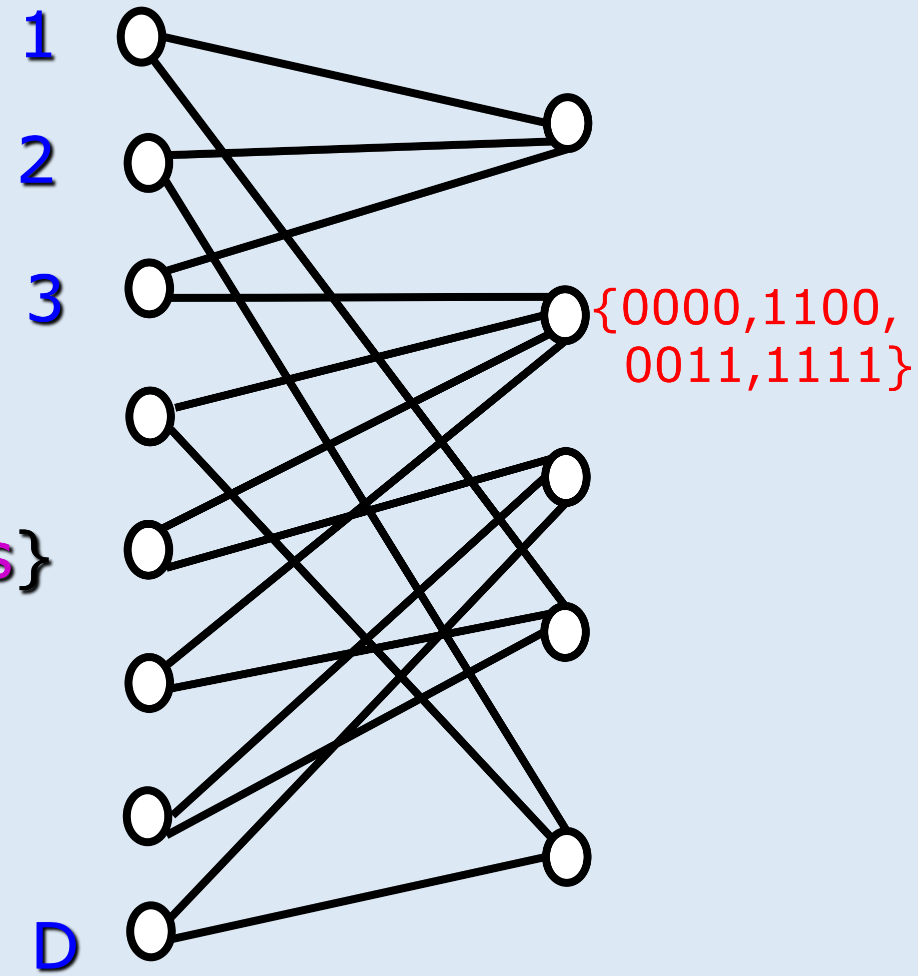
Algebraic Property = Code! (usually)

Basic Implications of Linearity [BHR]

- If P is linear, then:
  - Tester can be made non-adaptive.
  - Tester makes one-sided error
    - (f 2 P) tester always accepts).
- Motivates:
  - Constraints:
    - k-query test => constraint of size k:
      - value of f at $®_1$,… $®_k$ constrained to lie in subspace.
  - Characterizations:
    - If non-members of P rejected with positive probability, then P characterized by local constraints.
      - functions satisfying all constraints are members of P.

Pictorially

- f = assgm't to left
- Right = constraints
- Characterization of P:
  P = {f sat. all constraints}



Sufficient conditions?

- Linearity + k-local characterization
) k-local testability?

- [BHR] No!
  - Elegant use of expansion
  - Rule out obvious test; but also <u>any</u> test … of <u>any</u> "q(k)"-locality

- Why is characterization insufficient?
  - Lack of symmetry?

Example motivating symmetry

- Conjecture (AKKLR '96):
  - Suppose property P is a vector space over F_2;
  - Suppose its "invariant group" is "2-transitive".
  - Suppose P satisfies a k-ary constraint
    - **8** f **2** P, f(®$_1$) + ⋯ + f(®$_k$) = 0.

  - Then P is (q(k), **2**(k,δ),δ)-locally testable.

- Inspired by "low-degree" test over F_2. Implied all previous algebraic tests (at least in weak forms).

Invariances

- Property P invariant under permutation (function) ¼: D → D, if

  f 2 P) f o ¼ 2 P

- Property P invariant under group G if

  8 ¼ 2 G, P is invariant under ¼.

- Can ask: Does invariance of P w.r.t. "nice" G leads to local testability?

Invariances are the key?

- "Polling" works well when (because) invariant group of property is the full symmetric group.
- Modern property tests work with much smaller group of invariances.
- Graph property ~ Invariant under vertex renaming.
- Algebraic Properties & Invariances?

Abstracting Algebraic Properties

- [Kaufman & S.]

- Range is a field F and P is F-linear.
- Domain is a vector space over F (or some field K extending F).

- Property is invariant under affine (sometimes only linear) transformations of domain.

- "Property characterized by single constraint, and its orbit under affine (or linear) transformations."

Invariance, Orbits and Testability

- Single constraint implies many
  - One for every permutation ¼ 2 Aut(P):
    - "Orbit of a constraint C"

      = {C **o** ¼ | ¼ 2 Aut(P)}
- Extreme case:
  - Property characterized by single constraint + its orbit: "Single orbit feature"
    - Most algebraic properties have this feature.
    - W.l.o.g. if domain = vector space over small field.

Example: Degree d polynomials

- Constraint: When restricted to a small dimensional affine subspace, function is polynomial of degree d (or less).
    - #dimensions • d/(K - 1)

- Characterization: If a function satisfies above for every small dim. subspace, then it is a degree d polynomial.

- Single orbit: Take constraint on any one subspace of dimension d/(K-1); and rotate over all affine transformations.

Some results

- If P is affine-invariant and has k-single orbit feature (characterized by orbit of single k-local constraint); then it is $(k, \delta/k^3, \delta)$-locally testable.
  - Unifies previous algebraic tests (in weak form) with single proof.

Analysis of Invariance-based test

- Property P given by $®_1,\ldots,®_k$; V 2 F^k

- P = {f | f(A($®_1$)) … f(A($®_k$)) 2 V, 8 affine A:$K^n \to K^n$}

- Rej(f) = $Prob_A$ [f(A($®_1$)) … f(A($®_k$)) not in V]

- Wish to show: If Rej(f) < $1/k^3$,
  then δ(f,P) = O(Rej(f)).

BLR Analog

- $\mathrm{Rej}(f) = \Pr_{x,y}[\, f(x) + f(y) \neq f(x+y)] < {}^{2}$
- Define $g(x) = \mathrm{majority}_y\ \{\mathrm{Vote}_x(y)\}$,
  where $\mathrm{Vote}_x(y) = f(x+y) - f(y)$.
- Step 0: Show $\delta(f,g)$ small
- Step 1: **8** x, $\Pr_{y,z}[\mathrm{Vote}_x(y) \neq \mathrm{Vote}_x(z)]$ small.
- Step 2: Use above to show g is well-defined and a homomorphism.

BLR Analysis of Step 1

- Why is $f(x+y) - f(y) = f(x+z) - f(z)$, usually?

?	$f(z)$	$- f(x+z)$
$f(y)$	0	$-f(y)$
$- f(x+y)$	$-f(z)$	$f(x+y+z)$

Generalization

- g(x) = ¯ that maximizes, over A s.t. A(®$_1$) = x,
  Pr$_A$ [¯,f(A(®$_2$),…,f(A(®$_k$)) 2 V]

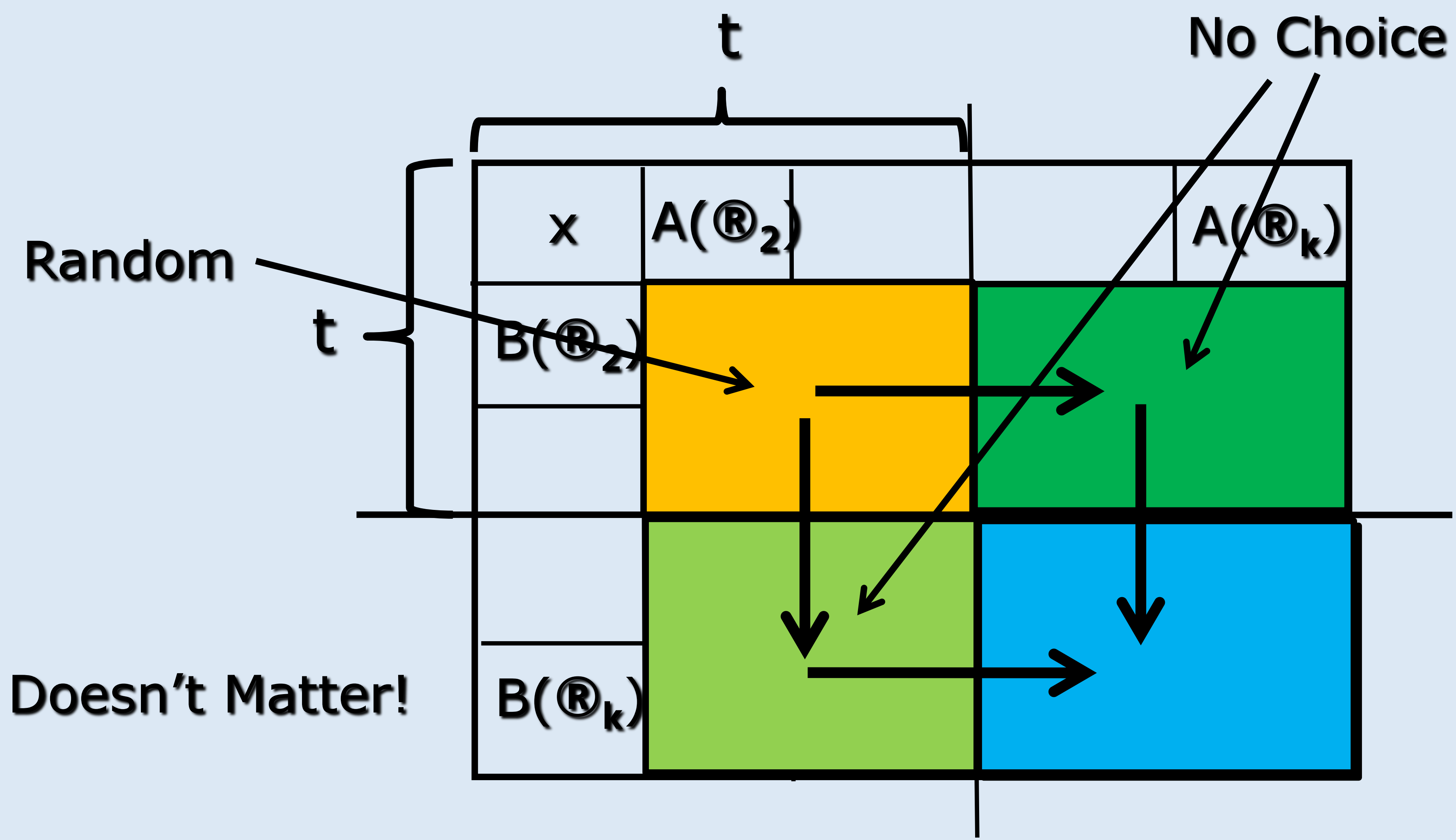
- Step 0: δ(f,g) small.

- Vote$_x$(A) = ¯ s.t. ¯, f(A(®$_2$))…f(A(®$_k$)) 2 V
  (if such ¯ exists)

- Step 1 (key): 8 x, whp Vote$_x$(A) = Vote$_x$(B).
- Step 2: Use above to show g 2 P.

Matrix Magic?

Say A(®$_1$) … A(®$_t$) independent; rest dependent



Some results

- If P is affine-invariant and has k-single orbit feature (characterized by orbit of single k-local constraint); then it is $(k, \delta/k^3, \delta)$-locally testable.
  - Unifies previous algebraic tests with single proof.

- If P is affine-invariant over K and has a single k-local constraint, then it is has a q-single orbit feature (for some $q = q(K,k)$)
  - (explains the AKKLR optimism)

Results (contd.)

- If P is affine-invariant over K and has a single k-local constraint, then it is has a q-single orbit feature (for some q = q(K,k))
- Proof Ingredients:
  - Analysis of all affine invariant properties.
  - Rough characterization of locality of constraints, in terms of degrees of polynomials in the family.
- Infinitely many (new) properties …

More details

- Understanding invariant properties:
  - Recall: all functions from K^n to F are Traces of polynomials
    - ($\mathrm{Trace}(x) = x + x^p + x^{p^2} + \ldots + x^{q/p}$ where $K = F_q$ and $F = F_p$)
  - If P contains $\mathrm{Tr}(3x^5 + 4x^2 + 2)$; then P contains $\mathrm{Tr}(4x^2)$ …
  - So affine invariant properties characterized by degree of monomials in family.
  - Most of the study … relate degrees to upper and lower bounds on locality of constraints.

Some results

- If P is affine-invariant over K and has a single k-local constraint, then it is has a q-single orbit feature (for some q = q(K,k))
    - (explains the AKKLR optimism)
- Unfortunately, q depends inherently on K, not just F … giving counterexample to AKKLR conjecture [joint with Grigorescu & Kaufman]
- Linear invariance when P is not F-linear:
    - Abstraction of some aspects of Green's regularity lemma … [Bhattacharyya, Chen, S., Xie]
    - Nice results due to [Shapira]

More results

- Invariance of some standard codes
  - E.g. "dual-BCH": Have k-single orbit feature! So are "more uniformly" testable.

    [Grigorescu, Kaufman, S.]
- Side effect: New (essentially tight) relationships between $Rej_{AKKLR}(f)$ and $\delta(f, \text{Degree-}d)$ over F_2 [with Bhattacharyya, Kopparty, Schoenebeck, Zuckerman]

More results (contd.)

- Invariance of some standard codes
- Side effect: New (essentially tight) relationships between $\mathrm{Rej}_{AKKLR}(f)$ and δ(f,Degree-d) over F_2

- One hope: Could lead to "simple, good locally testable code"?
  - (Sadly, not with affine-inv. [Ben-Sasson, S.])

- Still … other groups could be used? [Kaufman+Wigderson]

Conclusions

- Invariance seems to be a nice perspective on "property testing" …
  - Certainly helps unify many algebraic property tests.
  - But should be a general lens in sublinear time algorithmics.

Thanks