Sublinear Graph Approximation Algorithms

Krzysztof Onak
MIT

Motivation

- Want to learn a combinatorial parameter of a graph:
  - the maximum matching size
  - the independence number $\alpha(G)$,
  - the minimum vertex cover size,
  - the minimum dominating set size

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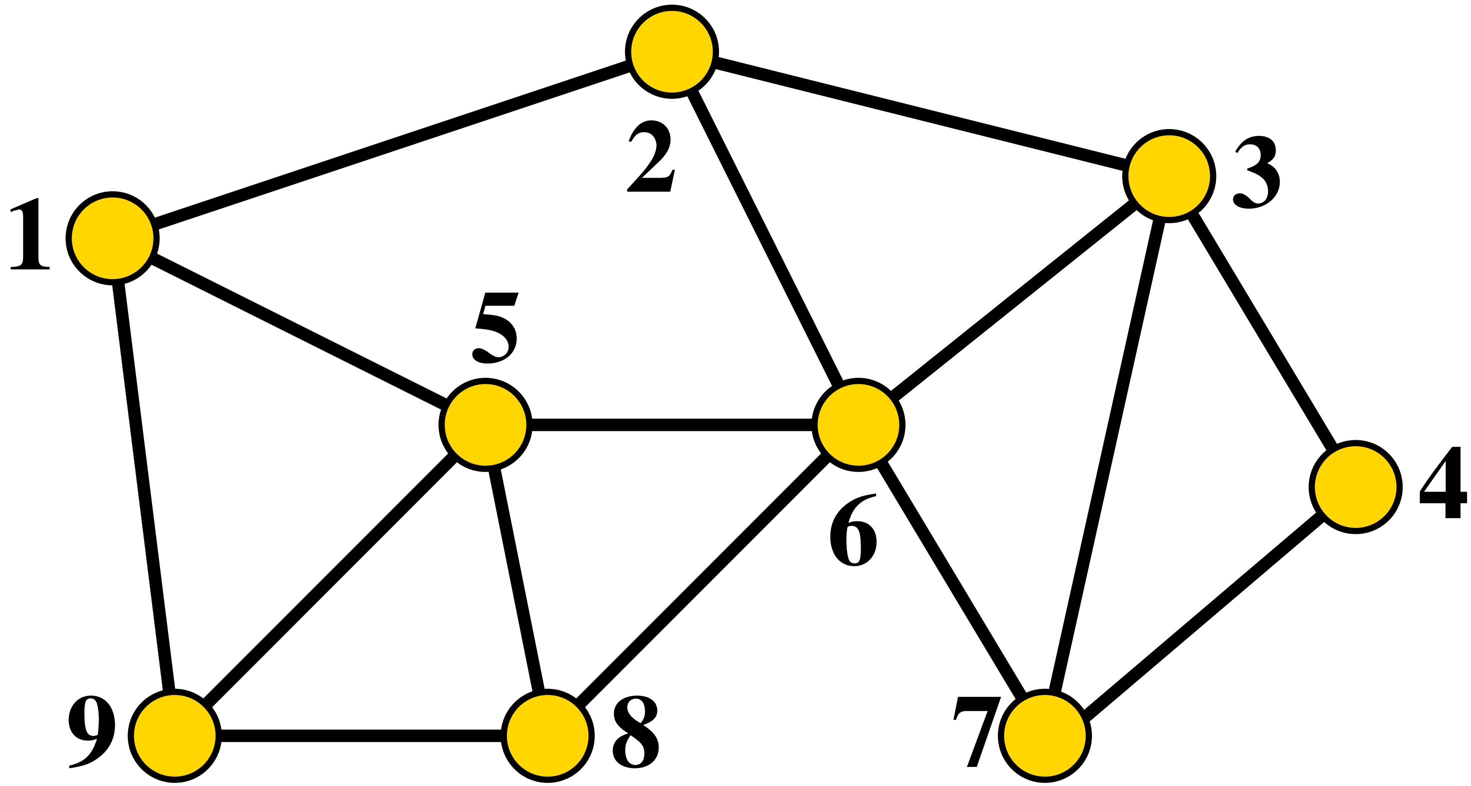
- The answer is YES in many cases

The Model

Bounded-degree graph G:



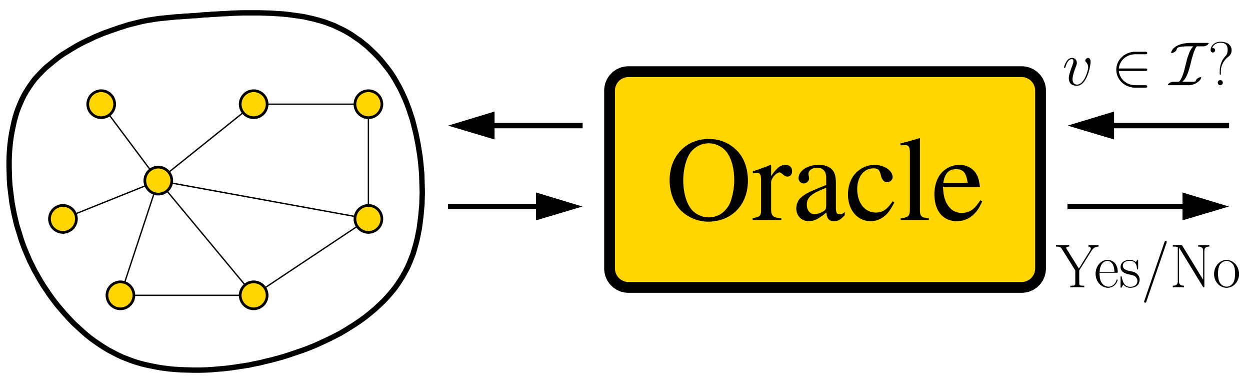
Query access to adjacency list of each node

Finding a Maximal Independent Set Locally

Oracle for Maximal Independent Set

Construct oracle $\mathcal{O}$:

- $\mathcal{O}$ has query access to $G = (V, E)$
- $\mathcal{O}$ provides query access to maximal independent set $\mathcal{I} \subseteq V$
- $\mathcal{I}$ is not a function of queries
  it is a function of G and random bits



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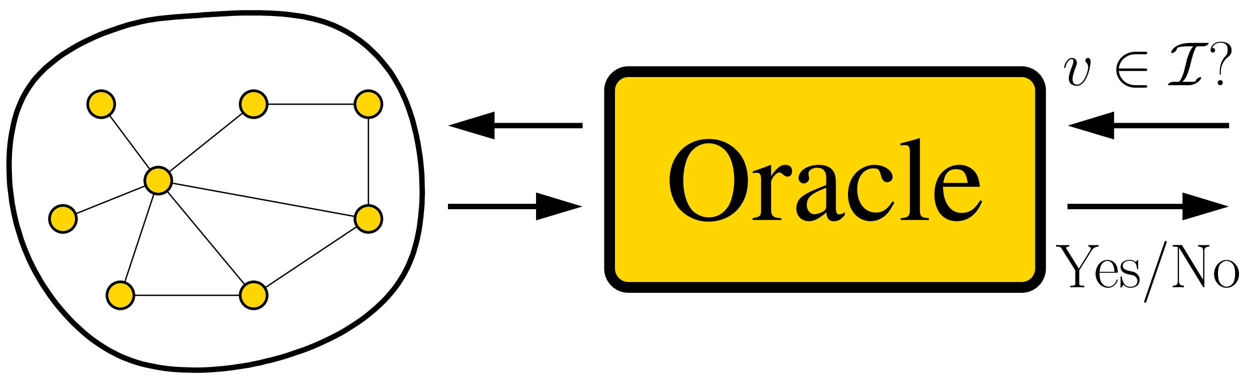

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One solution: Luby's maximal independent set algorithm (1986)
simulated locally [Marko, Ron 2007]

Here: a method better for sublinear algorithms

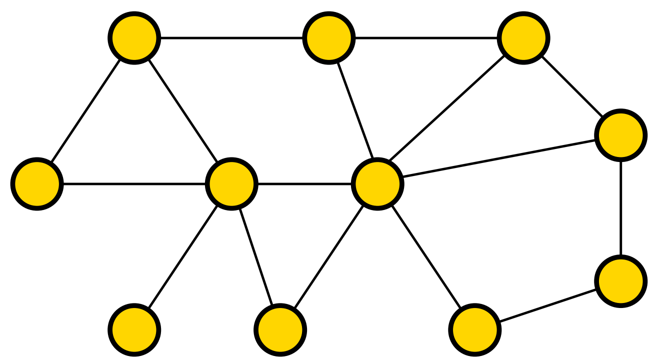
Our Method

Nguyen, O. (2008)

Main idea:

- select maximal independent set greedily
- consider vertices in random order

Random order $\equiv$ random numbers $r(v)$ assigned to each vertex



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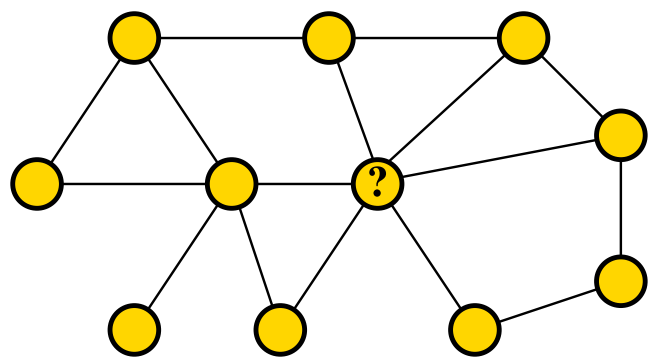

To check if $v \in \mathcal{I}$

- recursively check if neighbors w s.t. $r(w) < r(v)$ are in $\mathcal{I}$
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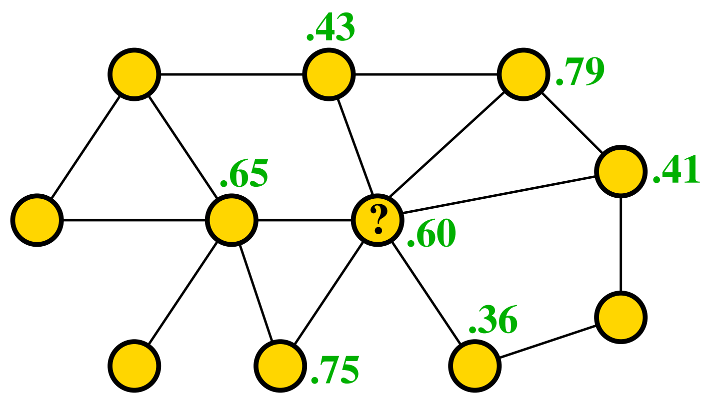

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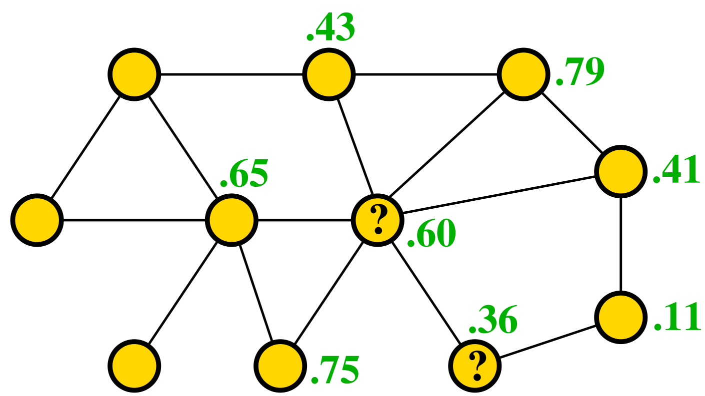

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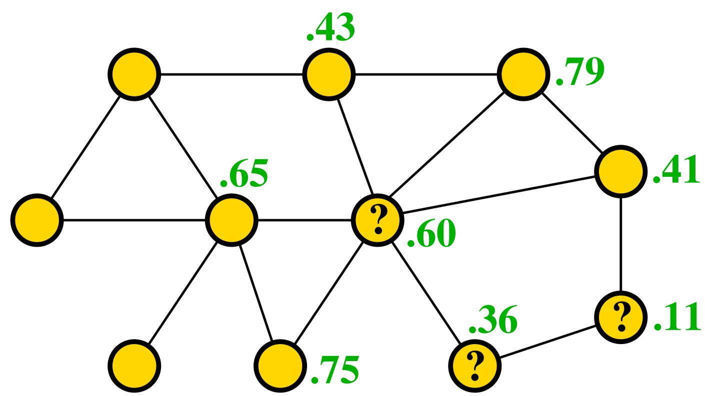

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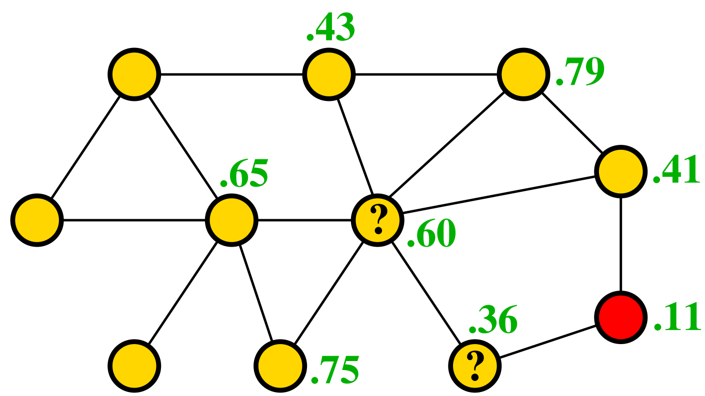

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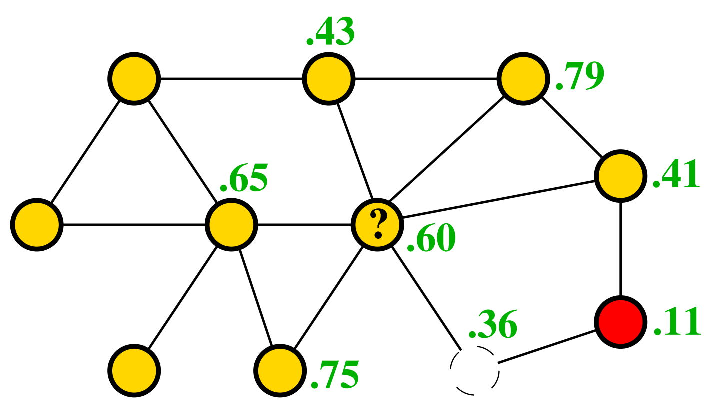

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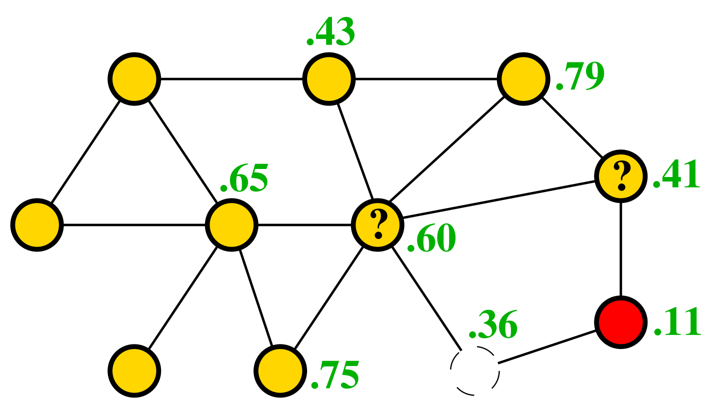

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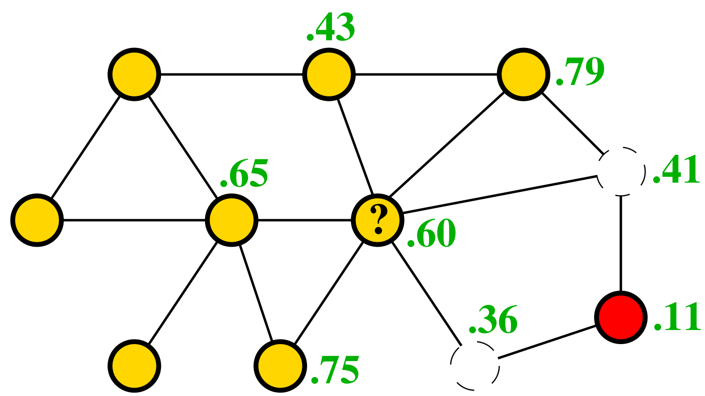

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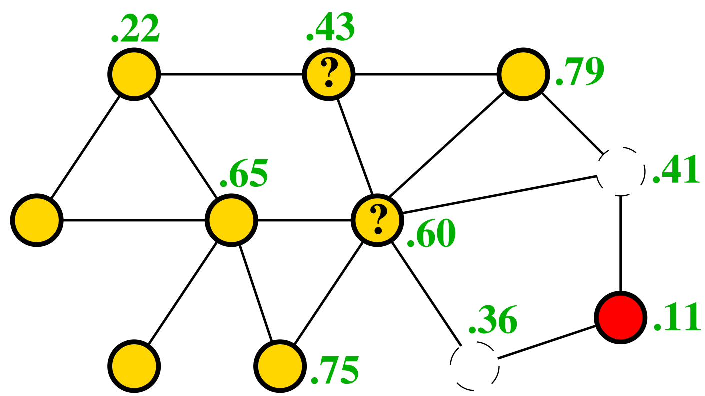

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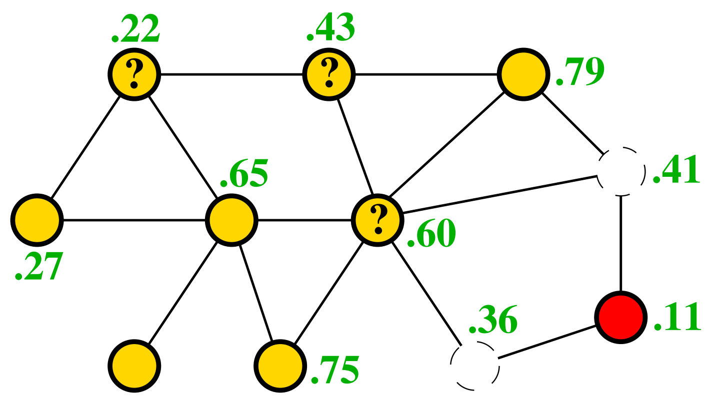

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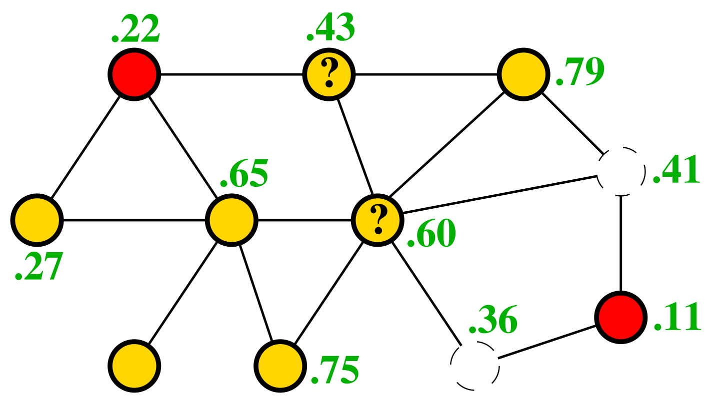

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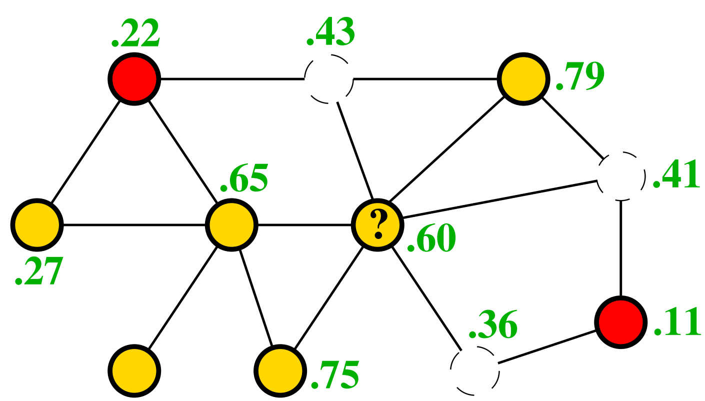

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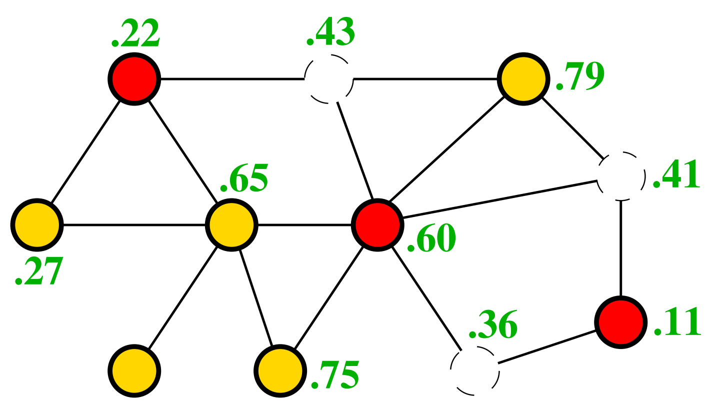

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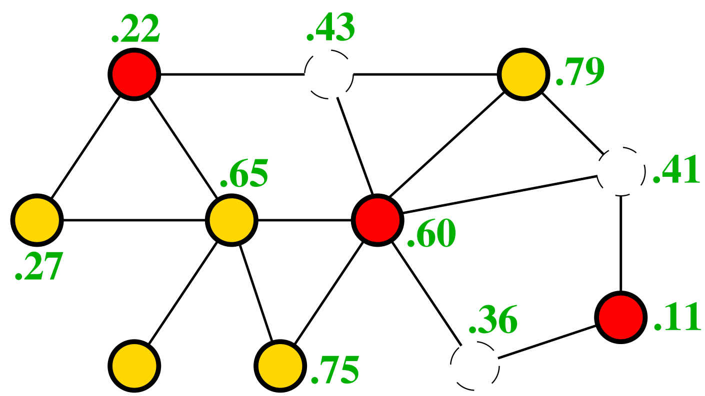

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$\mathrm{E}[$#visited vertices$]$ and query complexity of order $2^{O(d)}$

Improvement for Random Query

Yoshida, Yamamoto, Ito (STOC 2009)

Heuristic:

- Consider neighbors w of v in ascending order of $r(w)$
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Which gives:

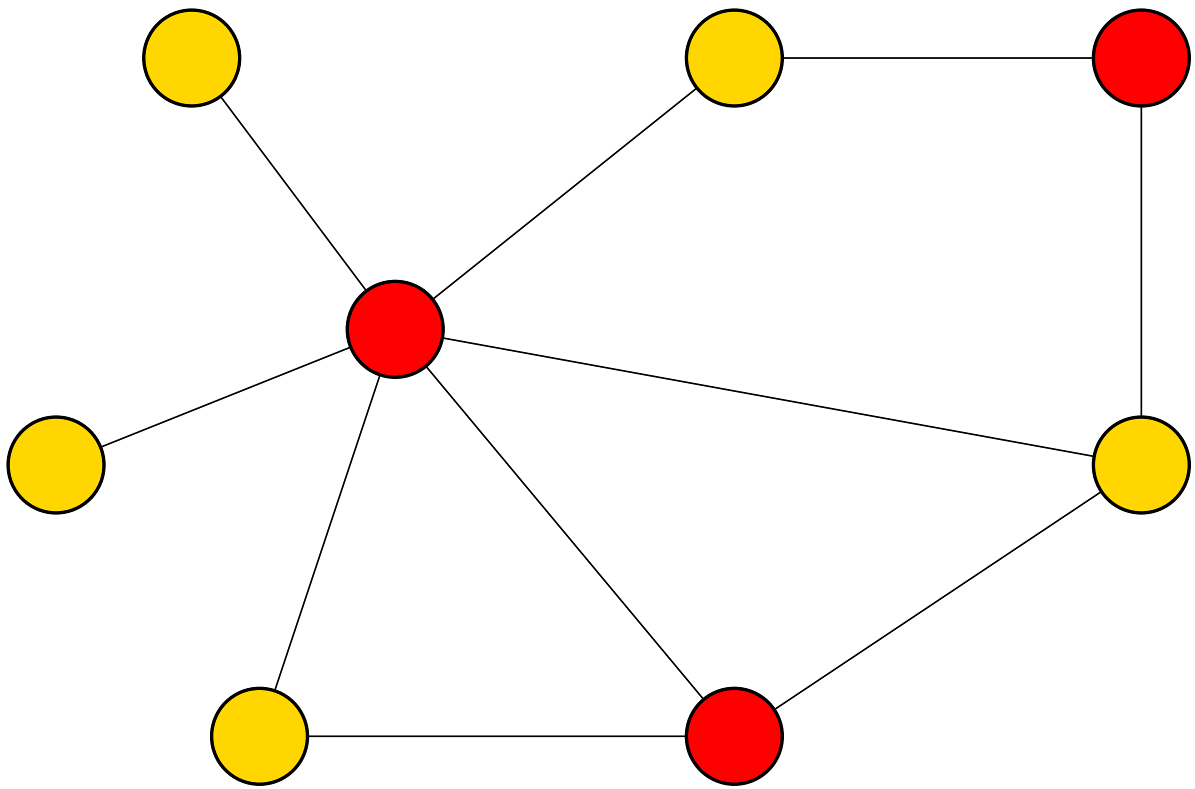
expected query complexity for random vertex = $O(d^2)$

Simplest Application: Vertex Cover

Vertex Cover

Graph $G = (V, E)$

Goal: find smallest set S of nodes such that each edge has endpoint in S



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- Greedily find a maximal matching M
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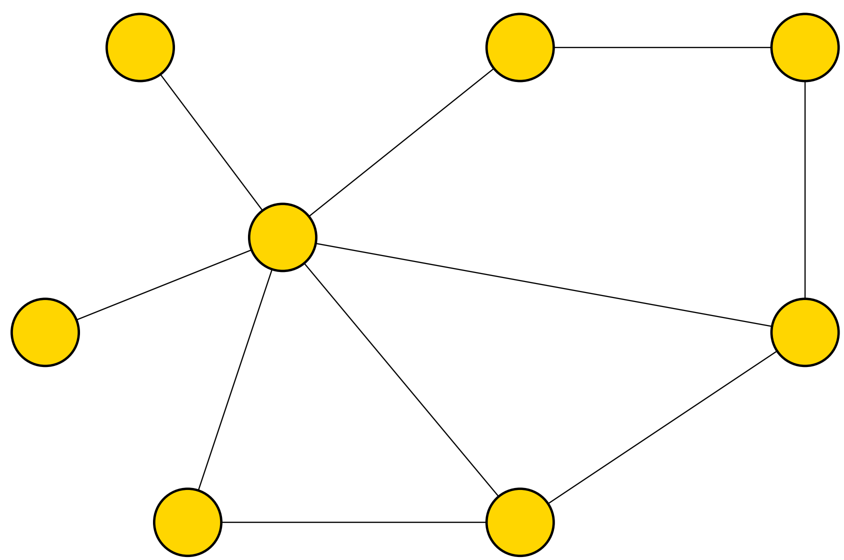

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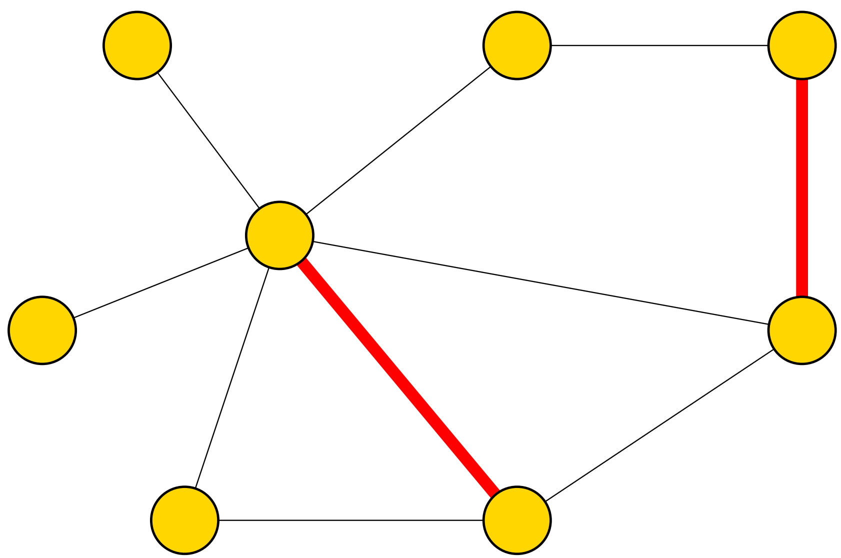

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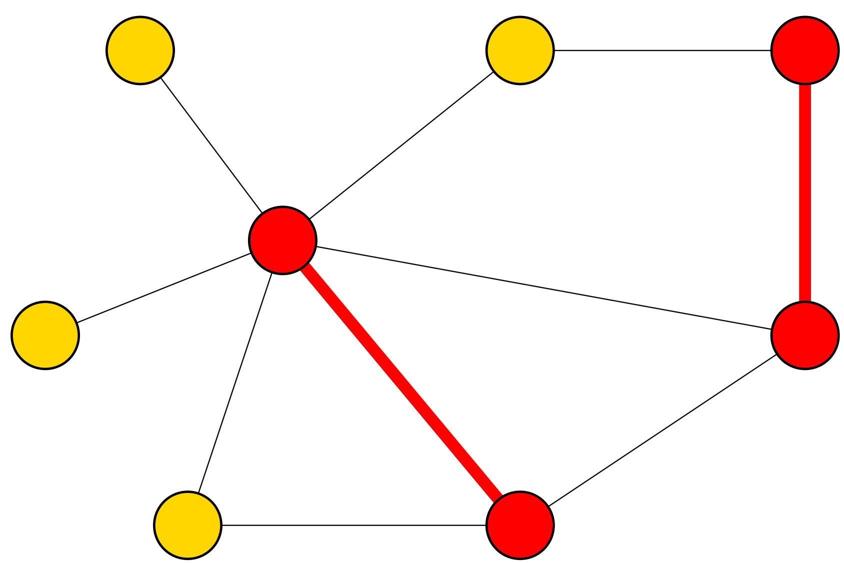

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Idea of Parnas and Ron (2007):

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$(2, \epsilon n)$-approximation:
Simulate $\mathcal{O}$ using our method

Query Complexity

Parnas, Ron (2007):

- oracles via simulation of local distributed algorithms
- used Kuhn, Moscibroda, Wattenhofer (2006)
- $\forall c > 2$, $(c, \epsilon n)$-approximation with $d^{O(\log(d))}/\epsilon^2$ queries
- $(2, \epsilon n)$-approximation with $d^{O(\log(d)/\epsilon^3)}$ queries

t communication rounds $\Rightarrow d^{O(t)}$ queries

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Yoshida, Yamamoto, Ito (2009) using our suggestion:

- $(2, \epsilon n)$-approximation with $O(d^3/\epsilon^2)$ queries

Lower Bounds

Trevisan 2007:

- $(c, \epsilon n)$-approximation requires $\Omega(\sqrt{n})$ queries for $c < 2$

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Parnas, Ron 2007:

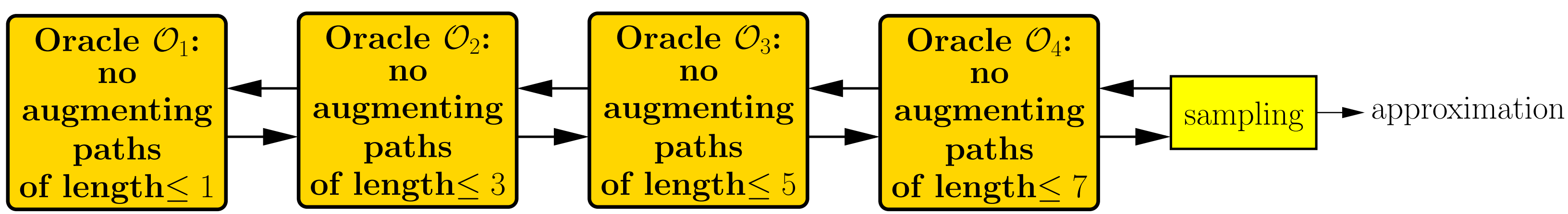
- $(O(1), \epsilon n)$-approximation requires $\Omega(d)$ queries

Other Problems

Maximum Matching Size

$(1, \epsilon n)$-approximation for maximum matching size

- Construct an oracle for a matching with no augmenting paths of length $\Theta(1/\epsilon)$
- Can be achieved by a sequence of oracles, where each oracle improves the matching from the previous oracle
- Each improvement corresponds to a maximal set of augmenting paths



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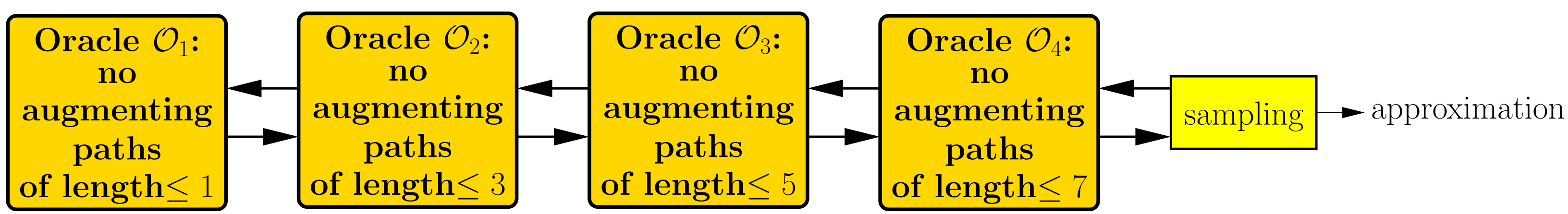

Query complexity and running time:

- Our analysis: $2^{d^{O(1/\epsilon)}}$
- Yoshida, Yamamoto, Ito (2009): $d^{O(1/\epsilon^2)}$

Set Cover and Dominating Set

Set Cover:

- Assumption:
  - each element in at most $t = O(1)$ of n sets
  - each set has at most $s = O(1)$ elements
- Guarantee: $(1 + \ln s, \epsilon n)$-approximation
- How: use classical greedy algorithm
- Complexity: function of s, t, and ϵ

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Dominating Set:

- $(1 + \ln(d+1), \epsilon n)$-approximation in time $\mathrm{function}(d, \epsilon)$

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- Alon: $\Omega(\log n)$ queries to $(o(\log d), \epsilon n)$-approximate

Maximum Matching

- Maximum Weight Matching:
  - Assumption: degree d and all weights in [0,1]
  - Guarantee: $(1, \epsilon n)$-approximation
  - How: use Pettie and Sanders (2004)
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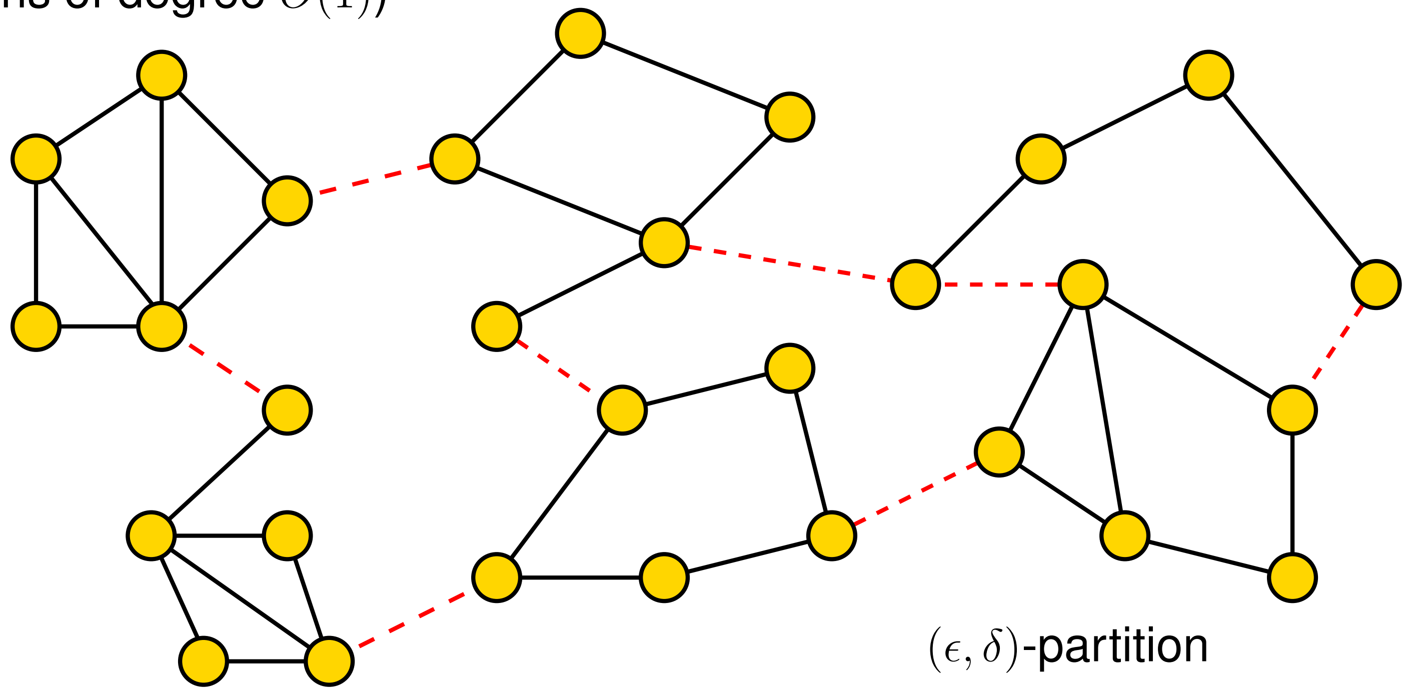
  - Guarantee: $(1, \epsilon n)$-approximation
  - How: use Pettie and Sanders (2004)
  - Complexity: function of d and ϵ
- Maximum Independent Set (Alon):
  - Upper bound:
    $\left(O\left(\frac{d \cdot \log\log d}{\log d}\right), \epsilon n\right)$-approximation in time $\mathrm{function}(d, \epsilon)$
  - Lower bound:
    $\Omega(\log n)$ queries to $\left(o\left(\frac{d}{\log d}\right), \epsilon n\right)$-approximate

Local Graph Partitions

[Hassidim, Kelner, Nguyen, O. 2009]

Hyperfinite Graphs

(All graphs of degree $O(1)$)

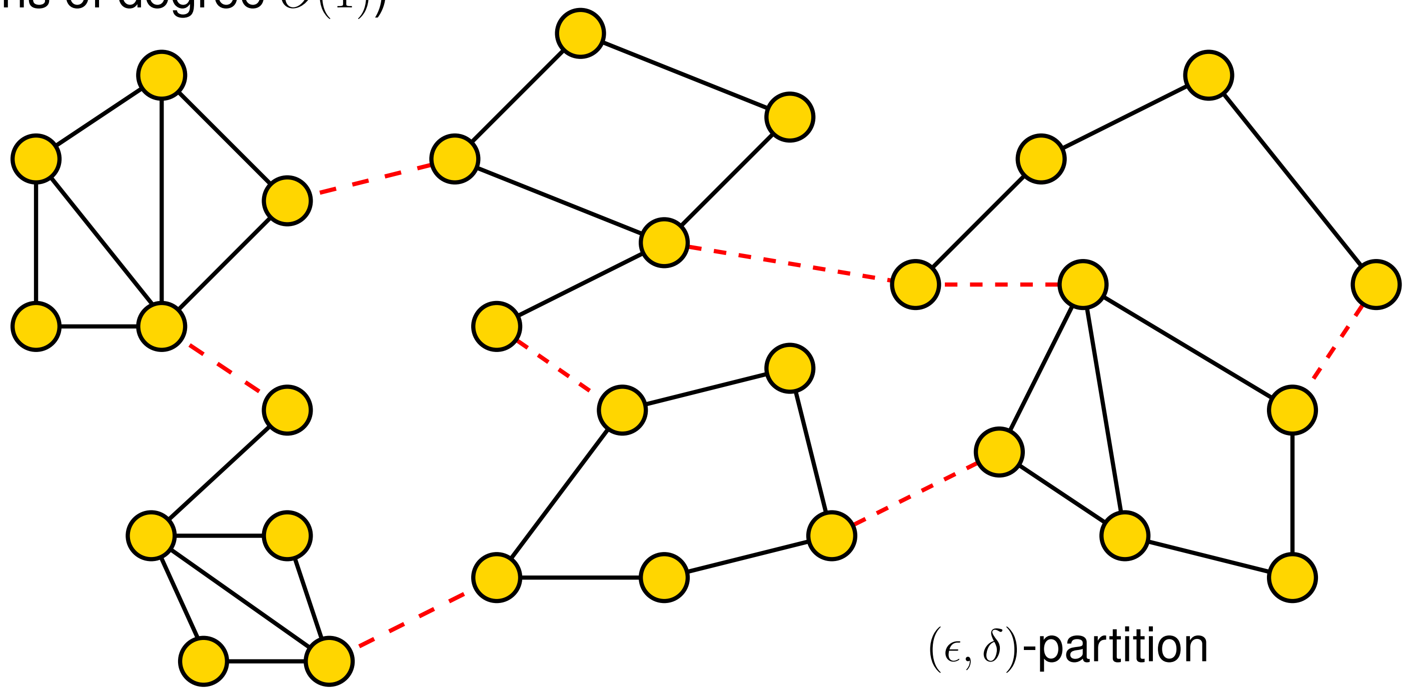


(ϵ, δ)-partition

- (ϵ, δ)-hyperfinite graphs: can remove $\epsilon|V|$ edges and get components of size at most δ

Hyperfinite Graphs

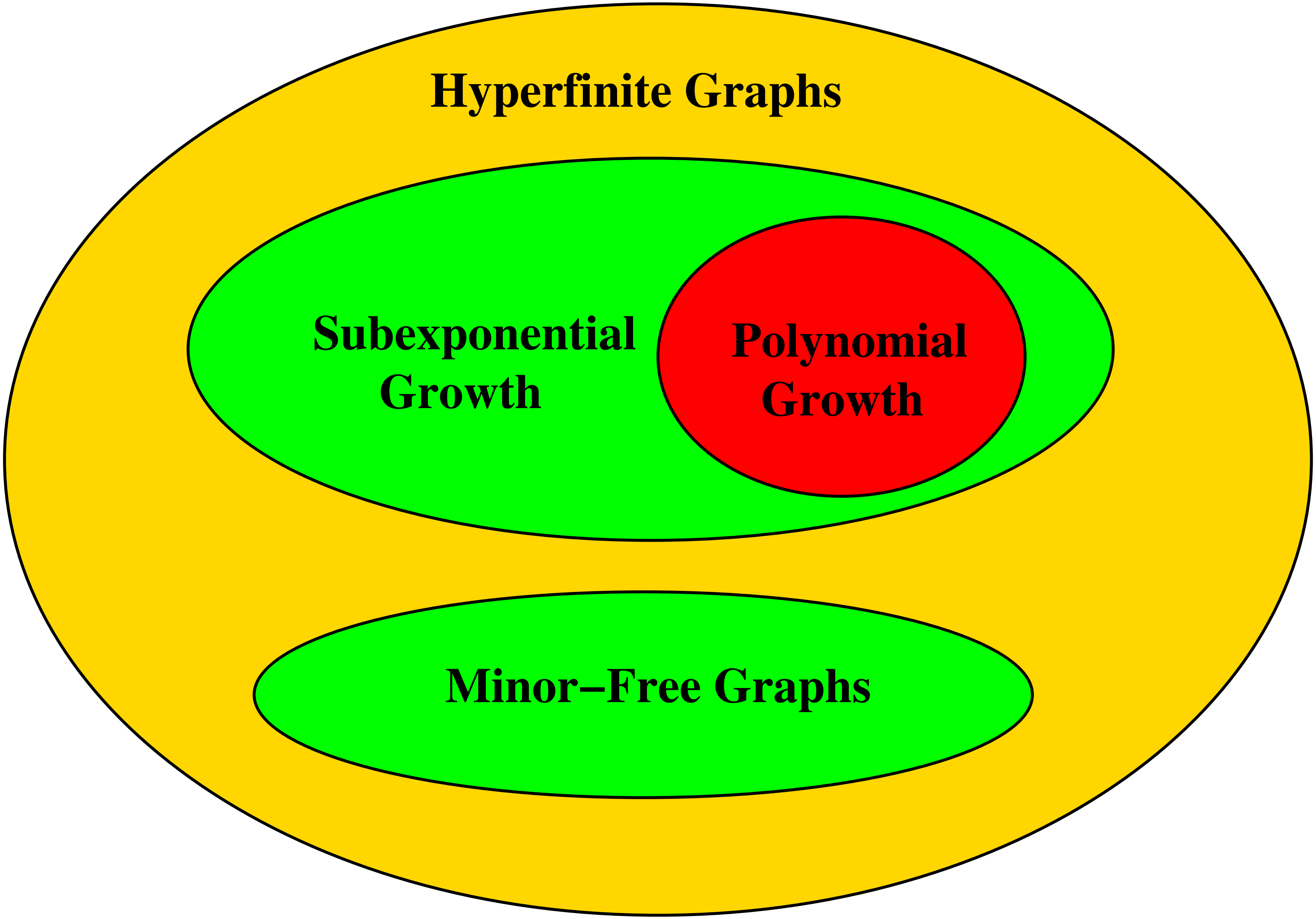
(All graphs of degree $O(1)$)



(ϵ, δ)-partition

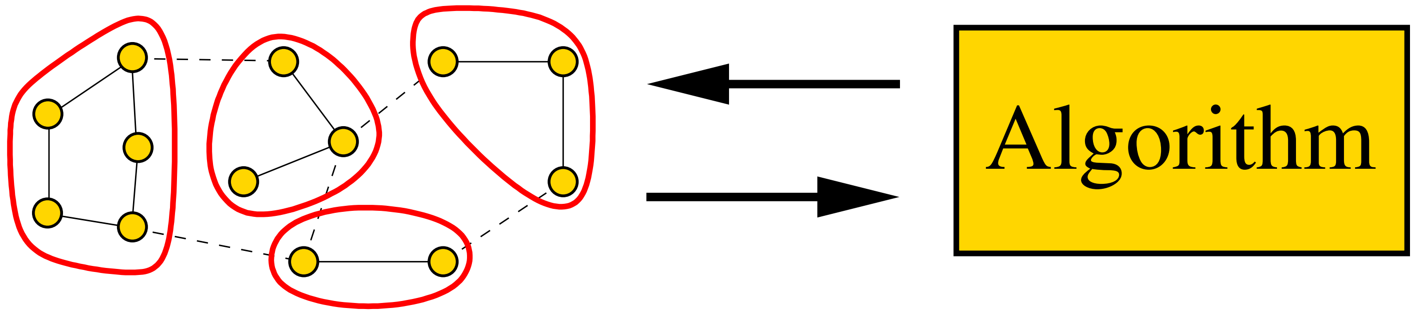
- (ϵ, δ)-hyperfinite graphs: can remove $\epsilon|V|$ edges and get components of size at most δ
- hyperfinite family of graphs: there is ρ such that all graphs are $(\epsilon, \rho(\epsilon))$-hyperfinite for all $\epsilon > 0$

Taxonomy


Hyperfinite Graphs
Subexponential Growth
Polynomial Growth
Minor–Free Graphs

Using a Partition

If someone gave us a $(\epsilon/2, \delta)$-partition:



- Sample $O(1/\epsilon^2)$ vertices
- Compute minimum vertex cover for the sampled components
- Return the fraction of the sampled vertices in the covers

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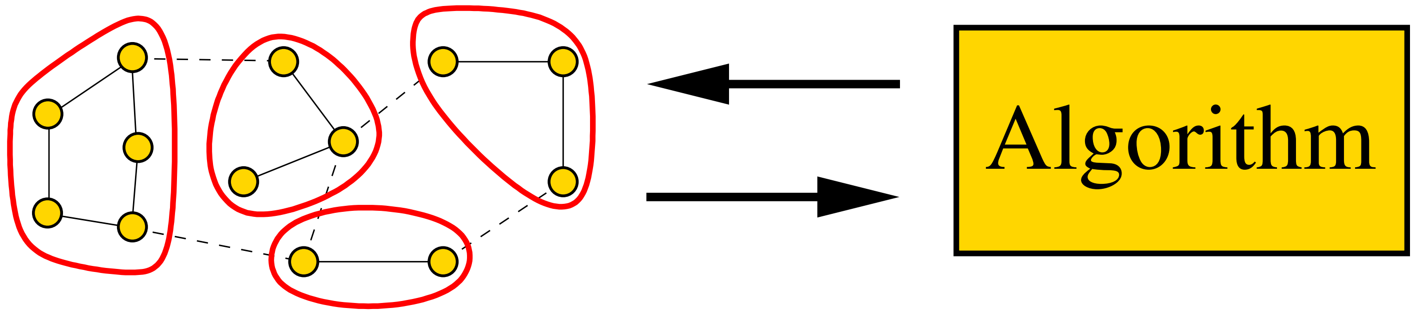

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This gives $\pm\epsilon$ approximation to $\mathrm{VC}(G)/n$ in constant time:

- Cut edges change $\mathrm{VC}(G)$ by at most $\epsilon n/2$
- Can compute vertex cover separately for each component

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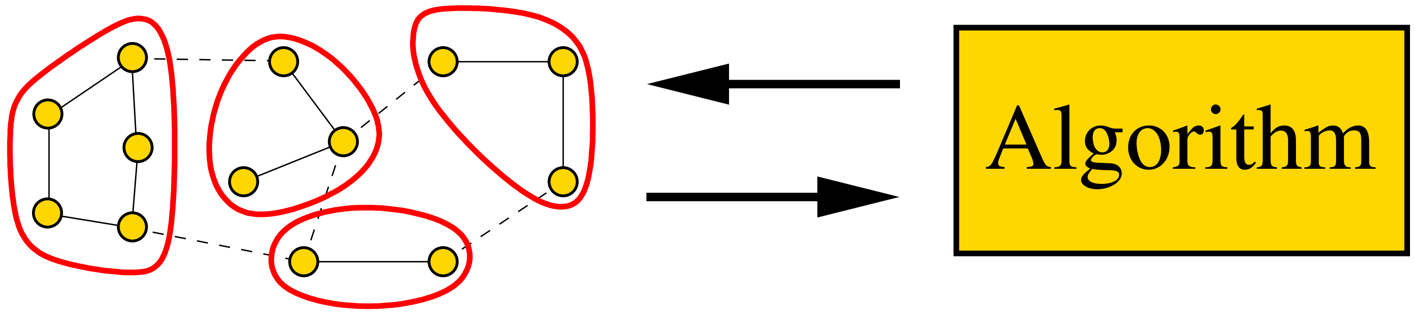

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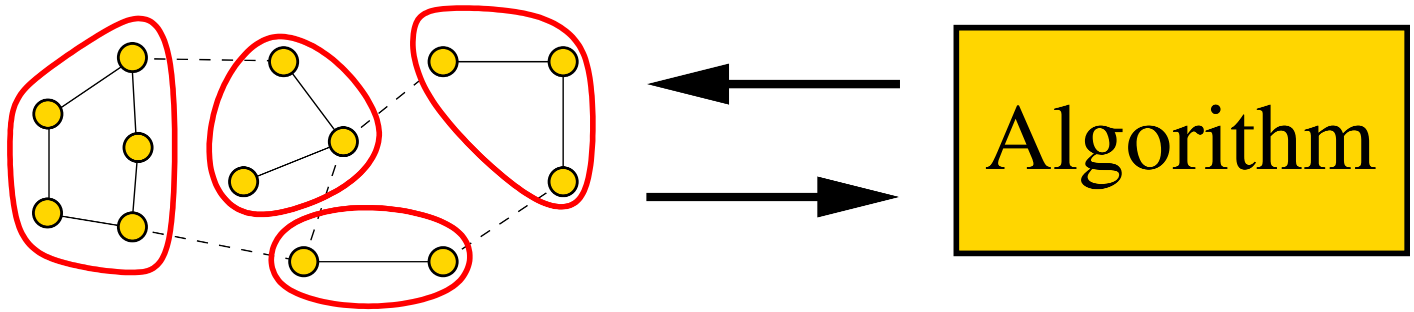

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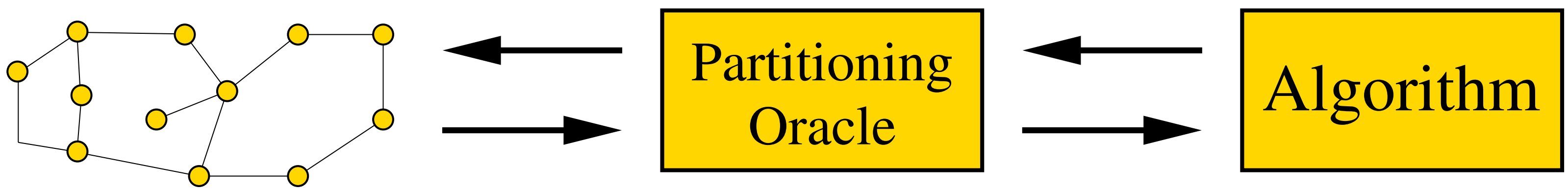

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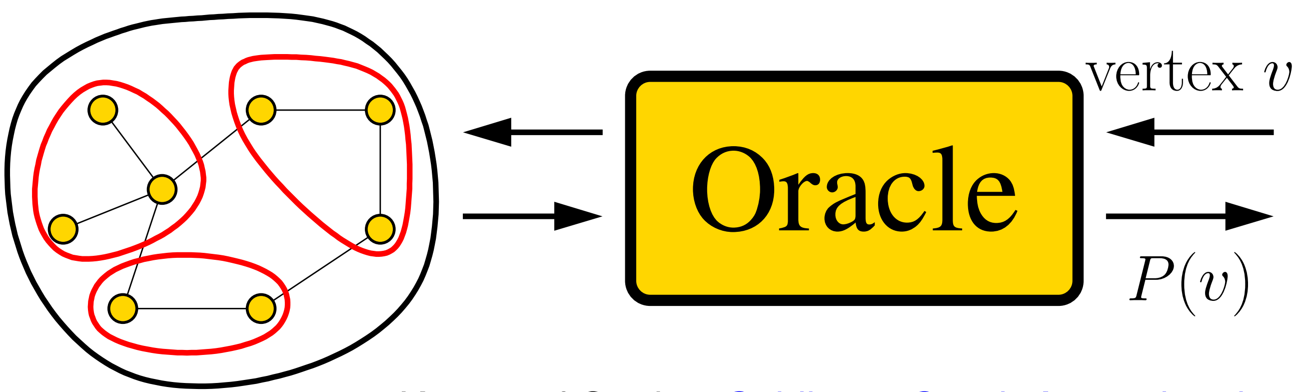
New Tool: Partitioning Oracles

Partitioning Oracle

$\mathcal{C} =$ fixed hyperfinite class

- oracle has query access to $G = (V, E)$
  (G need not be in $\mathcal{C}$)



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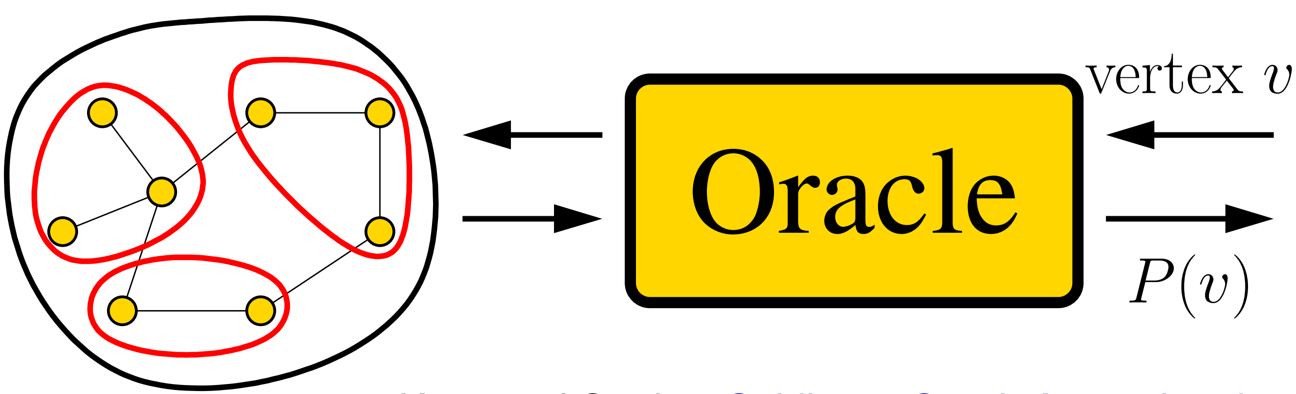

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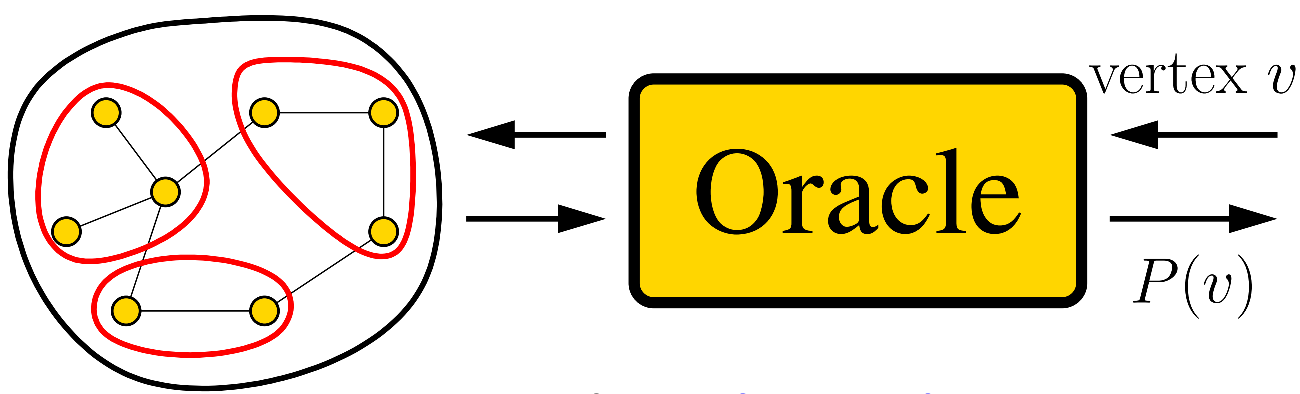

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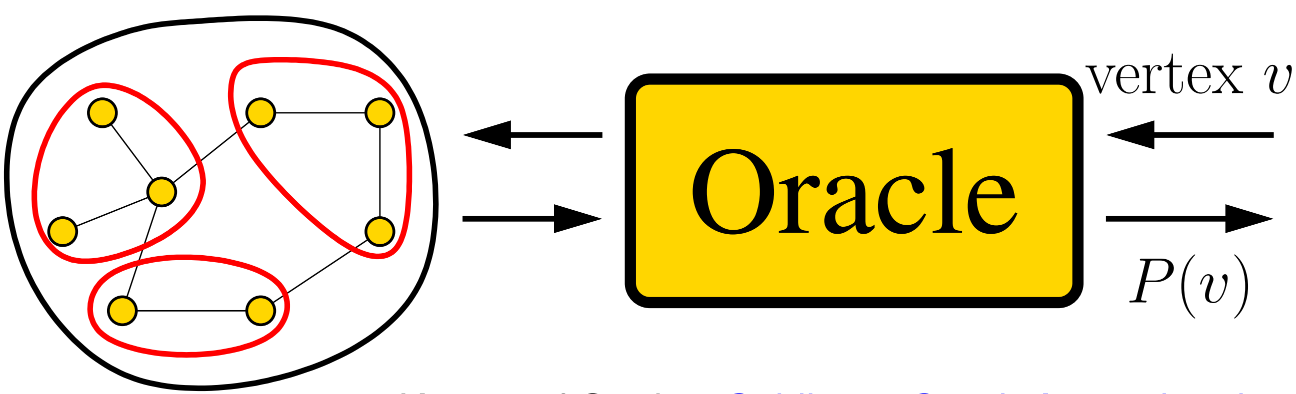

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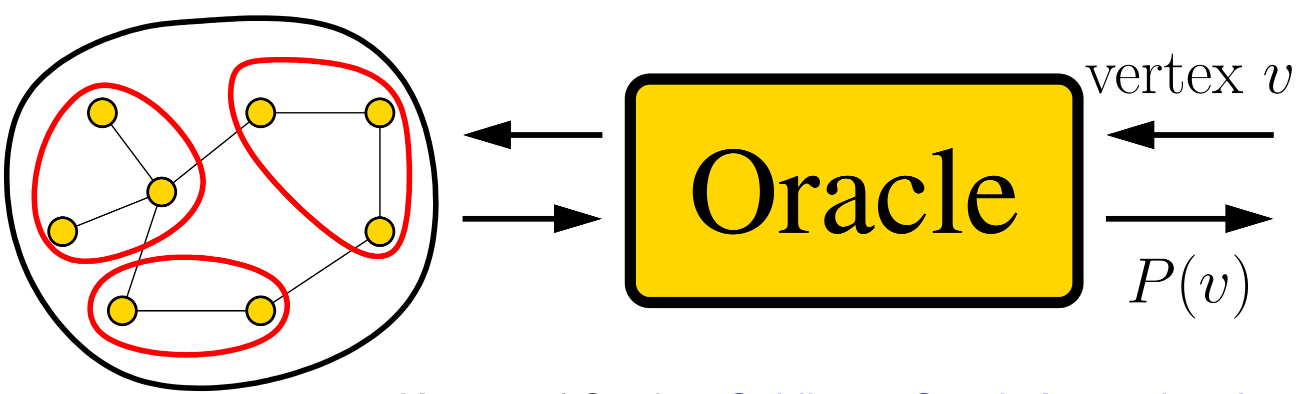
  - partition $P(\cdot)$ is not a function of queries,
    it is a function of graph structure and random bits



Our Oracles

- Generic oracle for any hyperfinite class of graphs
  - Query complexity: $2^{d^{O(\rho(\epsilon^3/54000))}}$
  - Via local simulation of a greedy partitioning procedure (uses [Nguyen, O. 2008])

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  - Via techniques from distributed algorithms [Czygrinow, Hańćkowiak, Wawrzyniak 2008]

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- Time complexity?
  - $Q =$ query complexity
  - $k =$ number of queries
  - Running time $= O(kQ \cdot \log(kQ))$

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3. Approximating distance to hereditary properties
   in hyperfinite graphs
   - Earlier only known to be testable [Czumaj, Shapira, Sohler 2009]

Application 1: Approximation

- For hyperfinite graphs, can get $\pm\epsilon n$ approximation to:
  - minimum vertex cover size
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    + Parnas, Ron (2007): for minor-free graphs

Application 2: Testing

Testing H-minor-freeness in the sparse graph model of Goldreich and Ron (1997)

- Input: query access to constant degree graph G
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- This work: $2^{\mathrm{poly}(1/\epsilon)}$ and simpler proof

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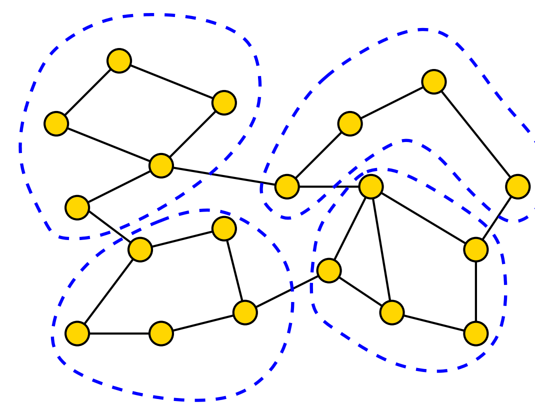

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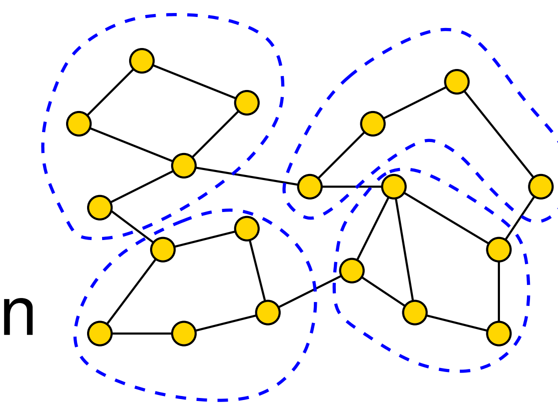
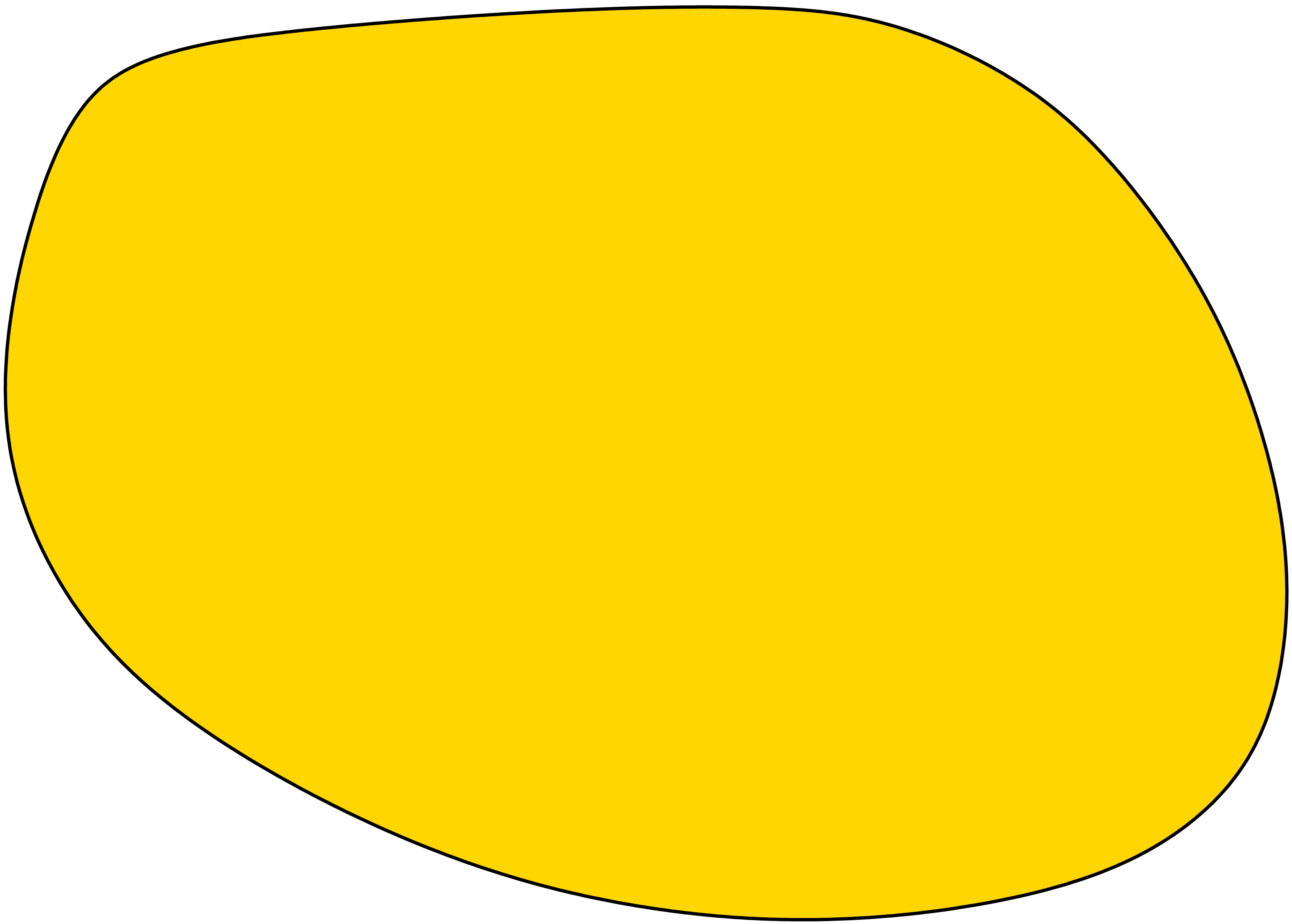


- Why it works:
  - planar: few edges cut in the partition
  - ϵ-far: either many edges cut
    or many copies of $K_{3,3}$ or K_5

Simplest Oracle

Iterative Procedure

Global procedure:



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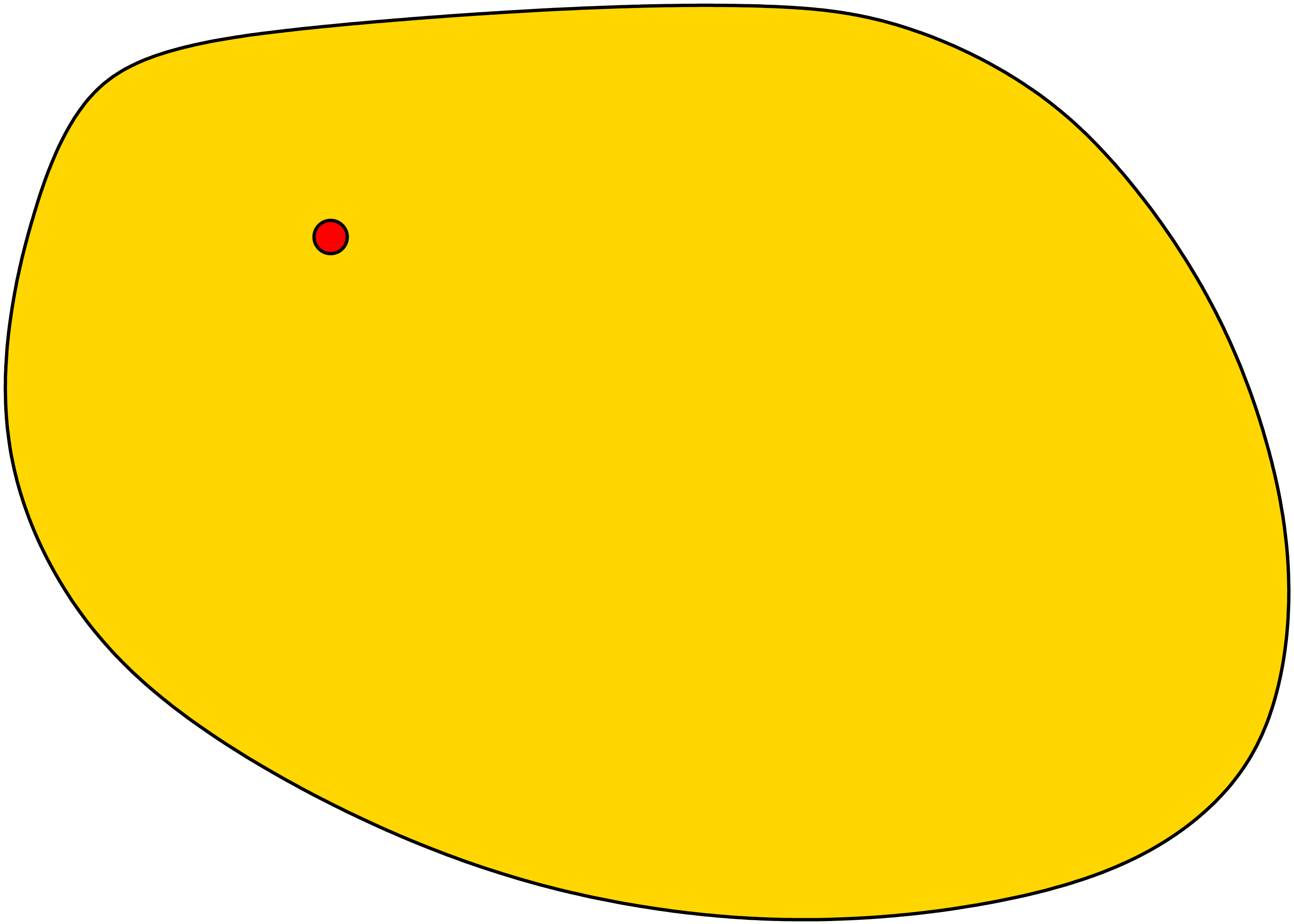

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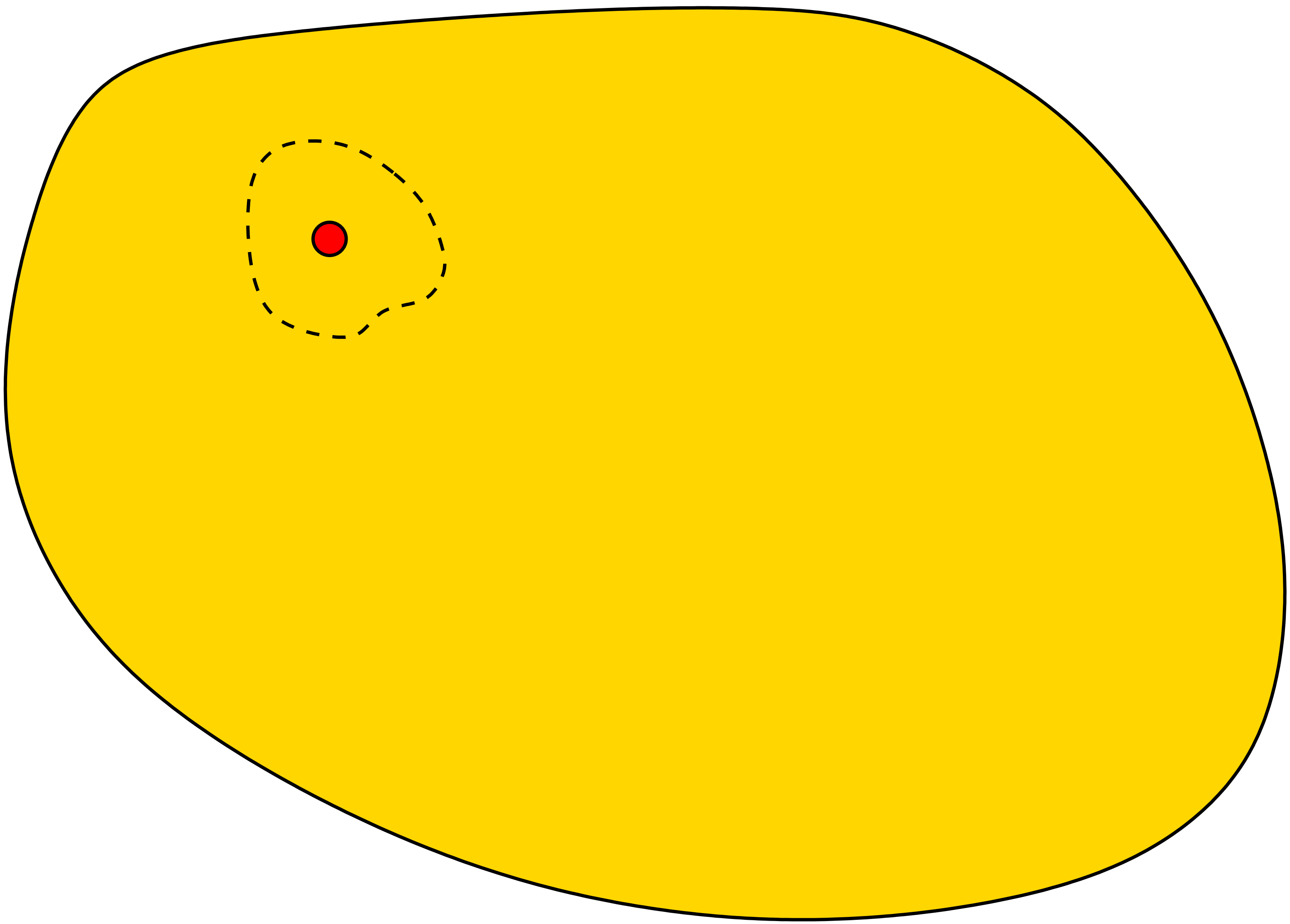

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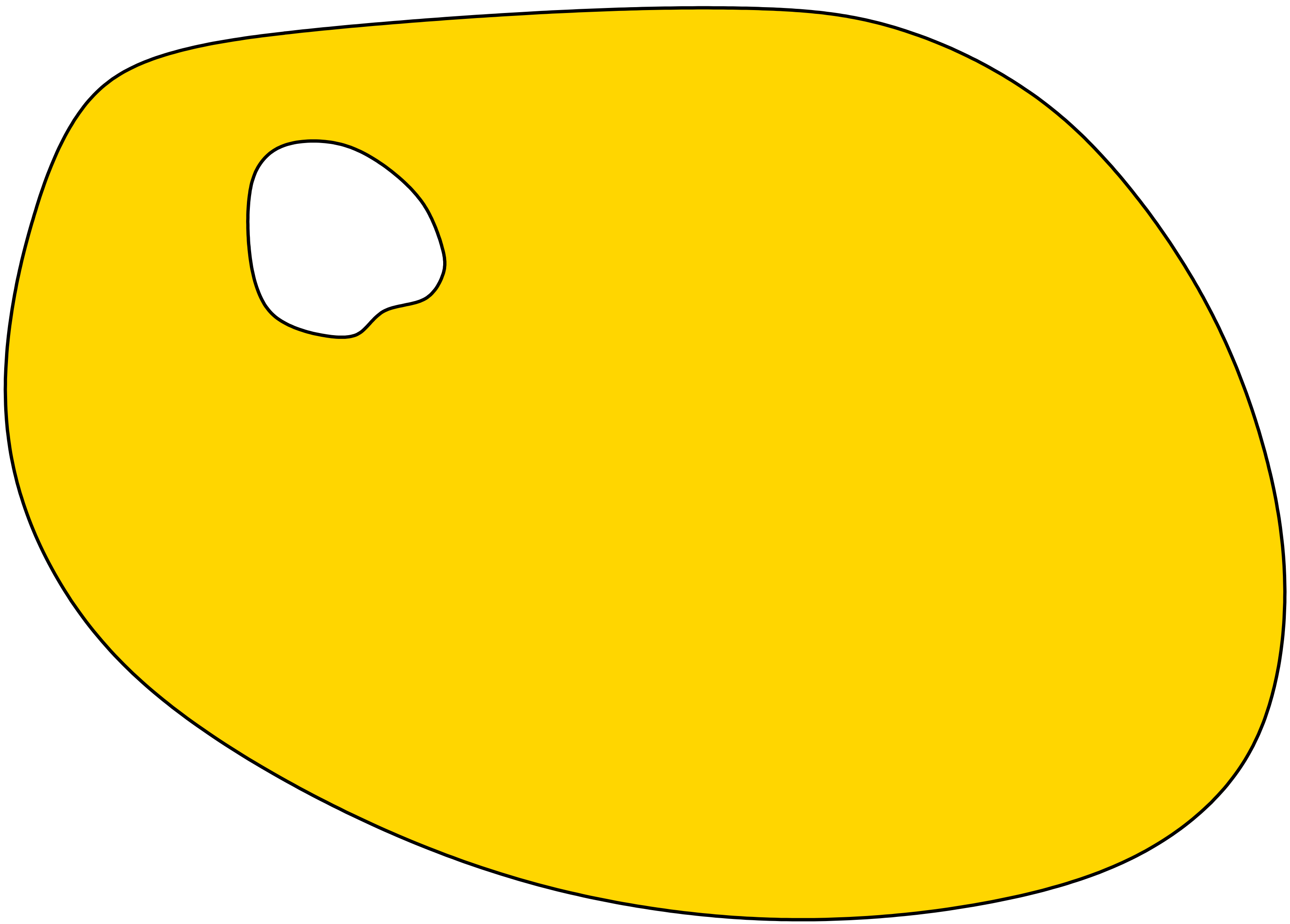

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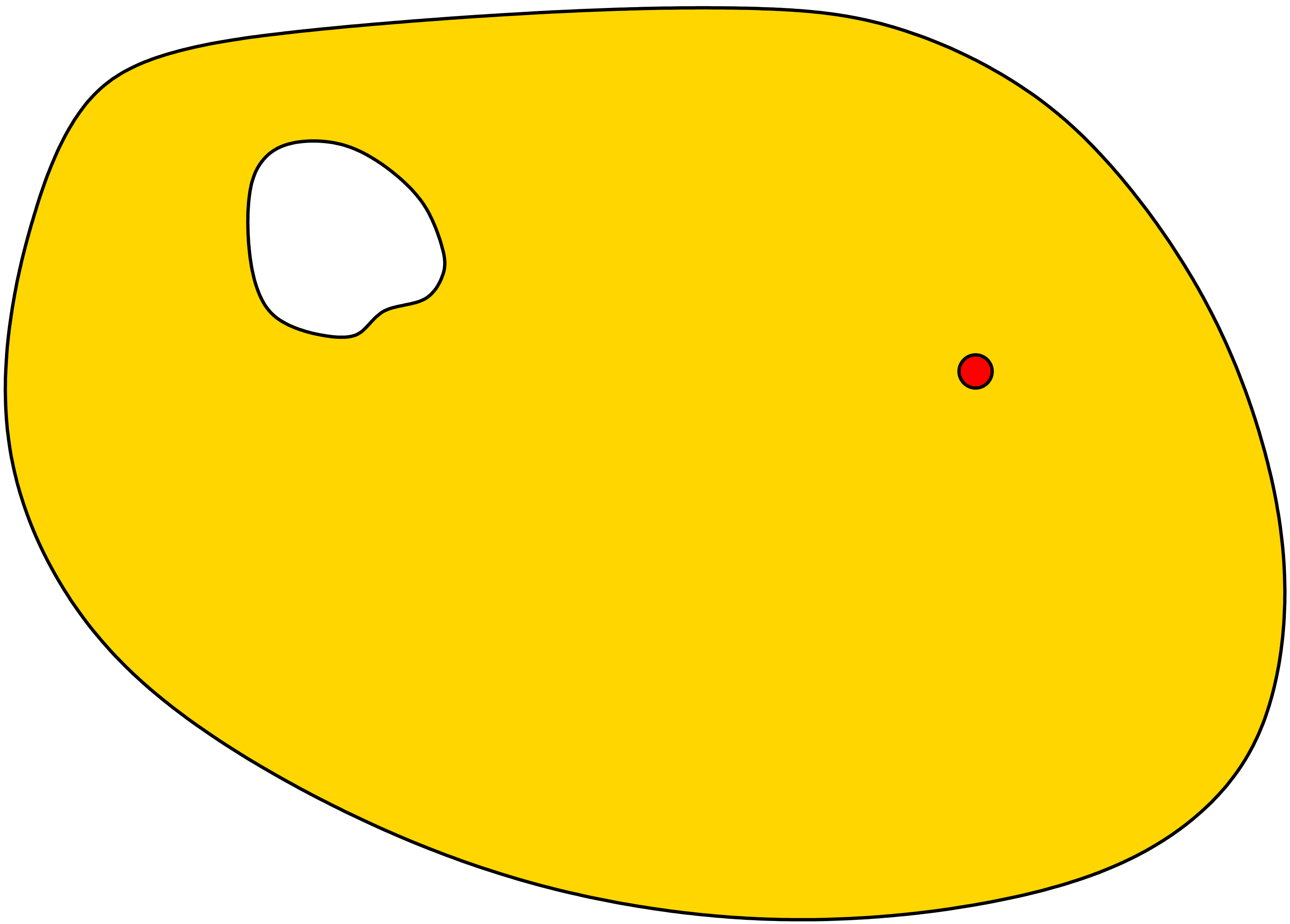

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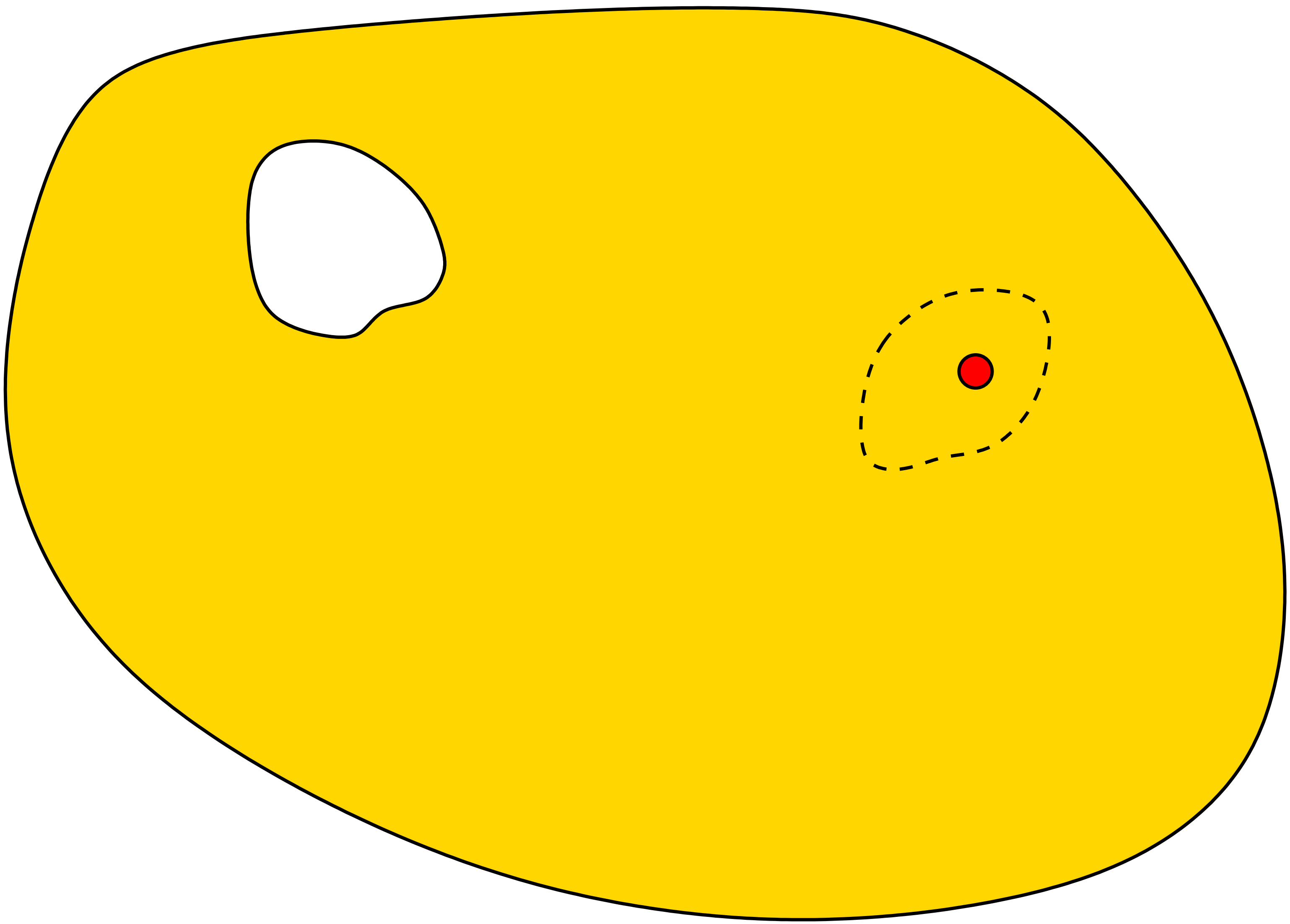

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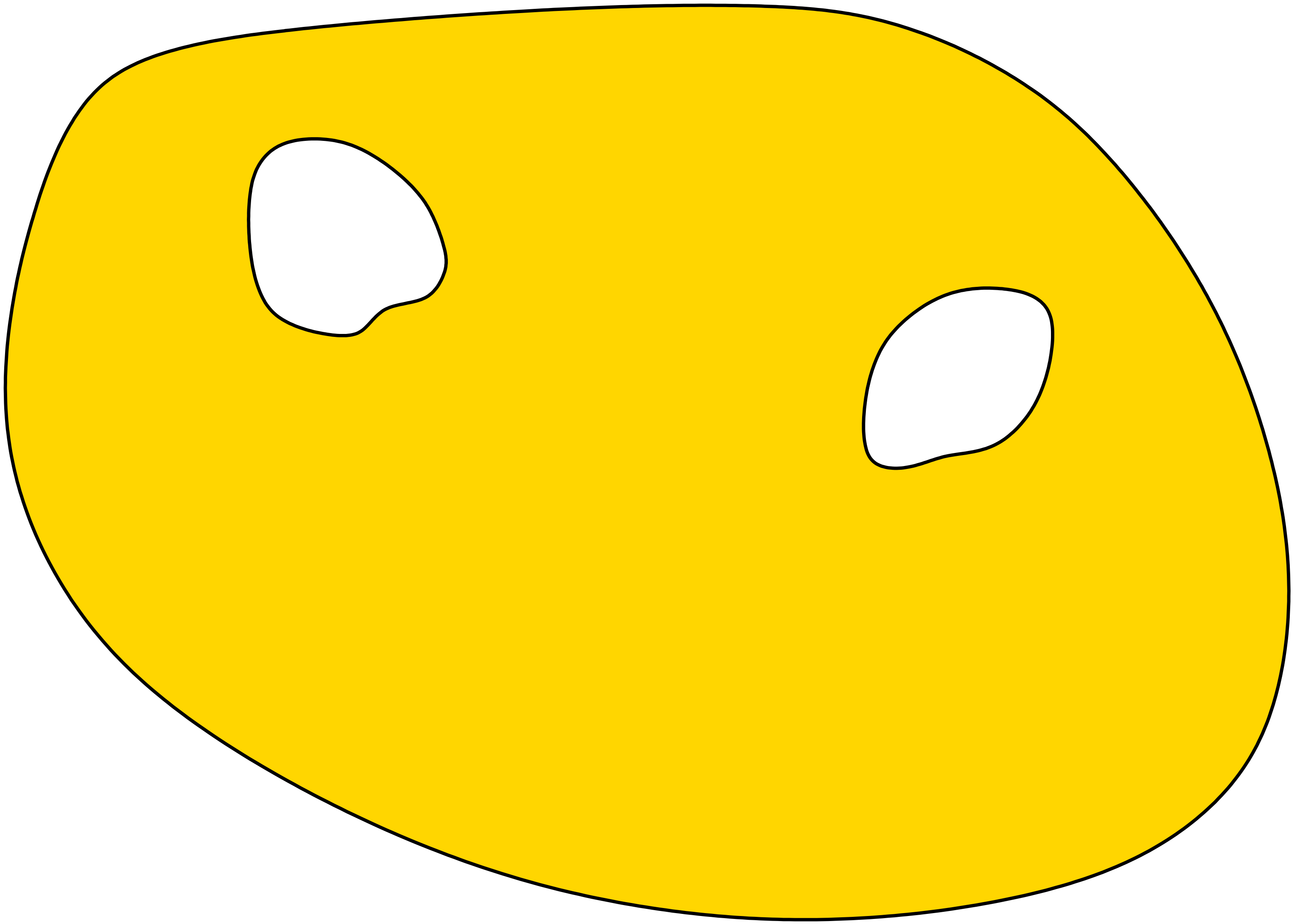

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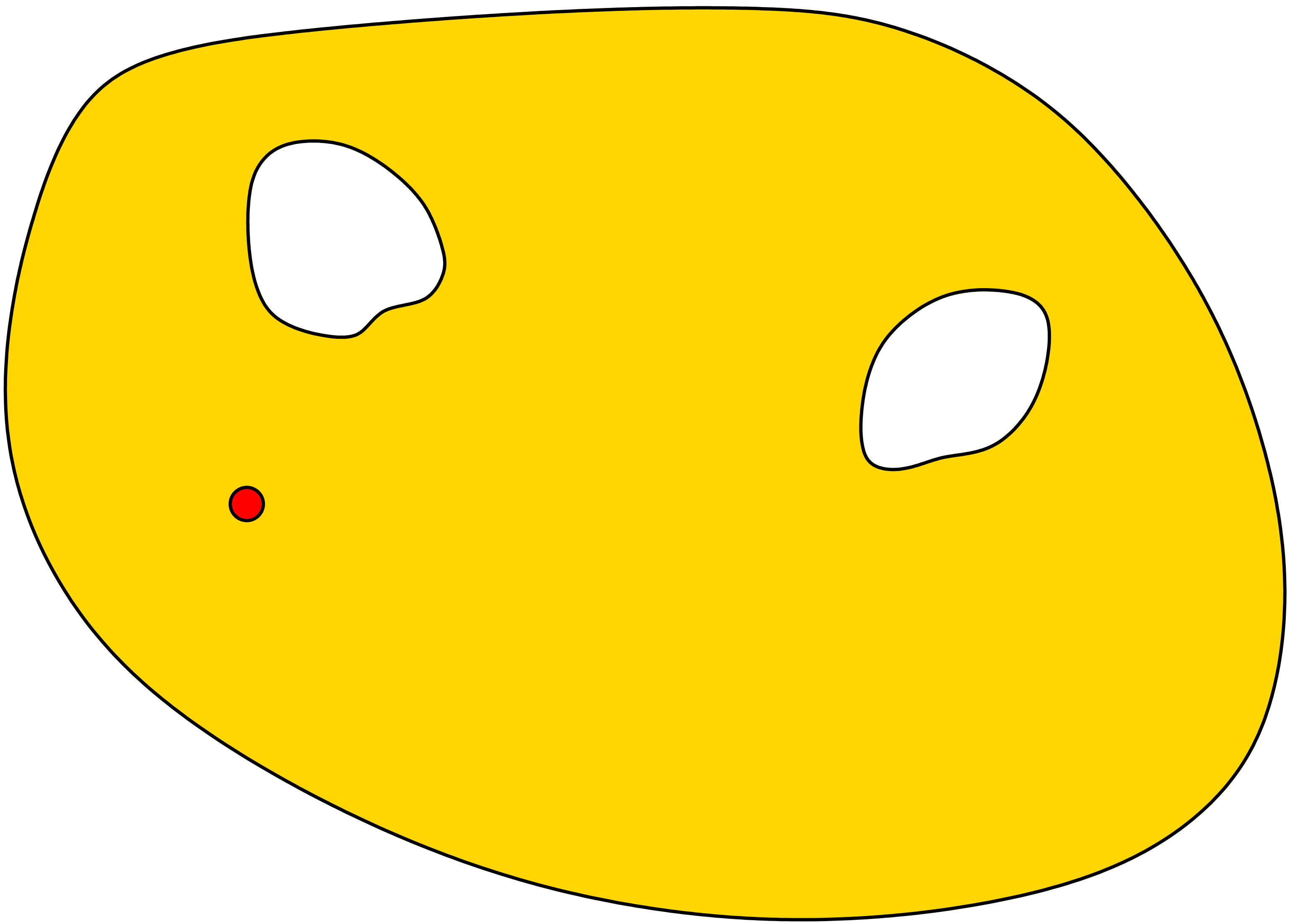

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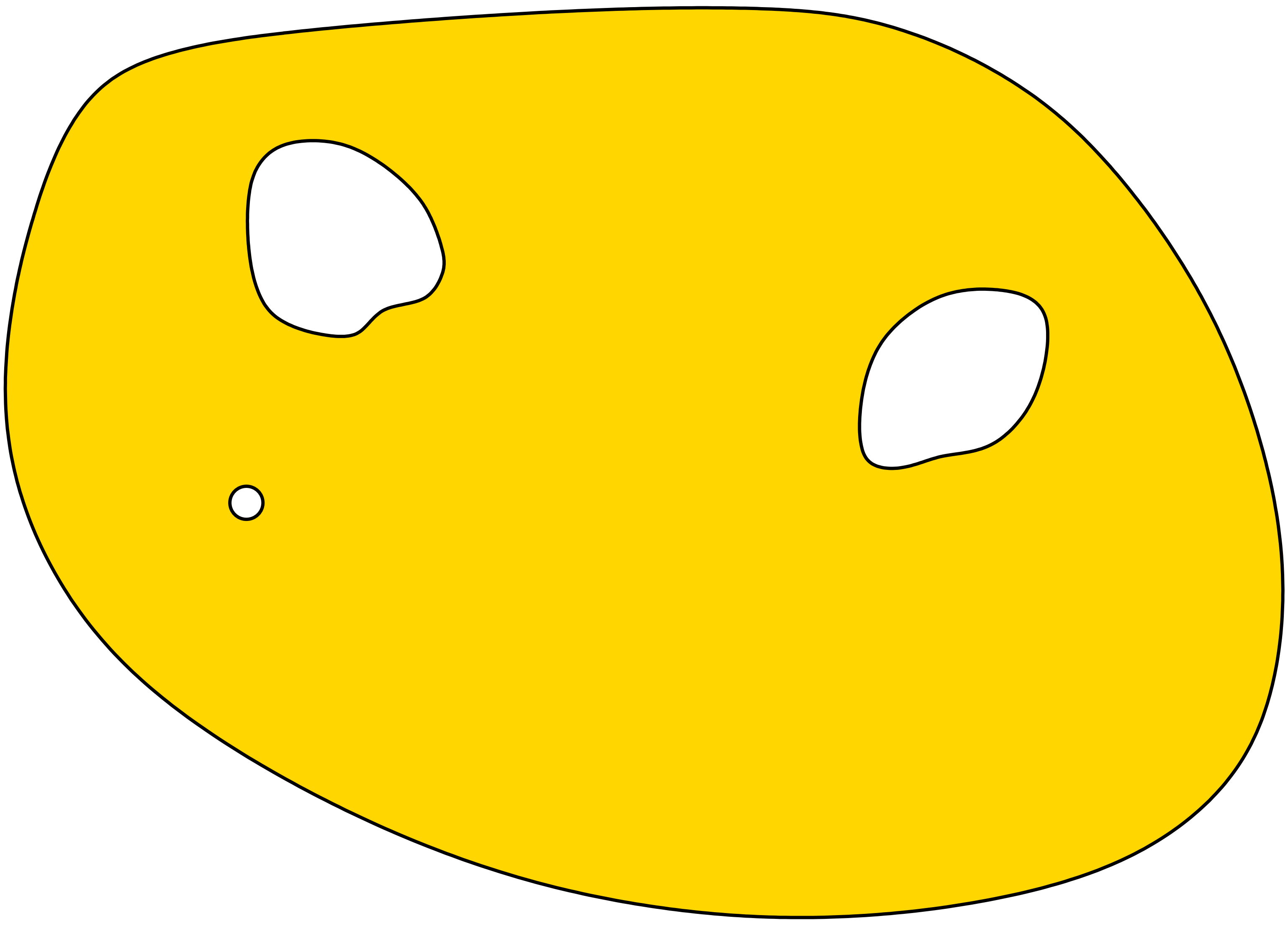
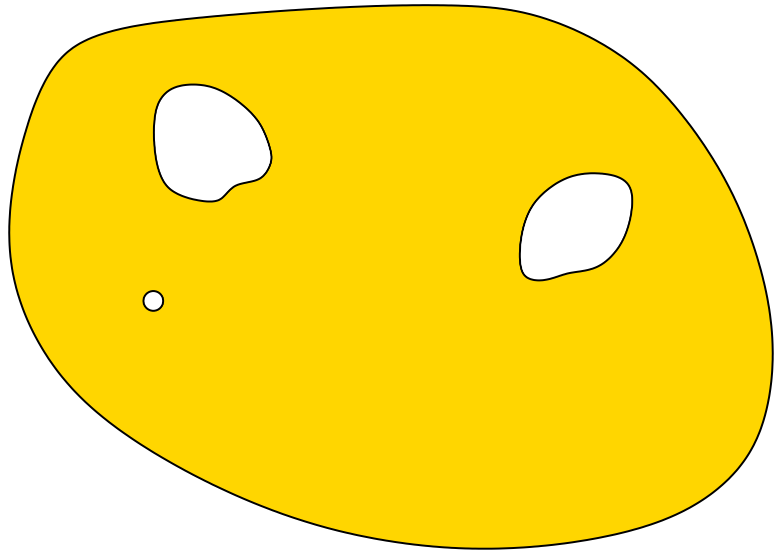


Local simulation



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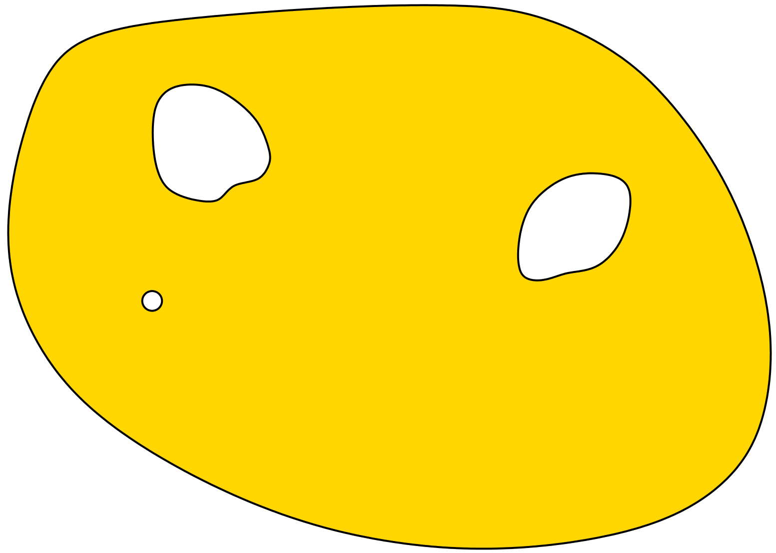

Use technique of Nguyen and O. (2008):

- Random numbers assigned to vertices generate a random permutation
- To find a component of v:
  - recursively check what happened for close vertices with lower numbers
  - if v still in graph, try to construct a component

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- Good approximation algorithms for other popular classes of graphs