Testing linear-invariant non-linear properties

Victor Chen
joint with Arnab Bhattacharyya, Madhu Sudan, Ning Xie

Tsinghua University

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  $$\mathcal{F} \subseteq \{\{0,1\}^n \to \{0,1\}\}$$

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Theorem (Green)

The property of triangle-freeness is locally testable.

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- design of algorithm: natural
- analysis of algorithm: Fourier-analytic, Green's regularity lemma

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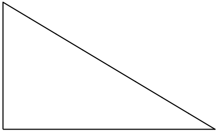

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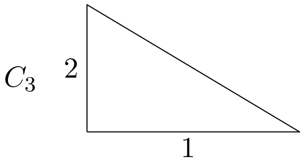

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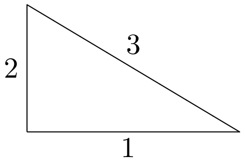

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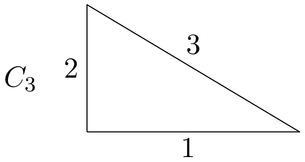

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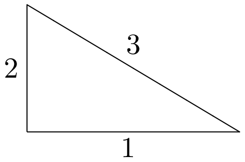

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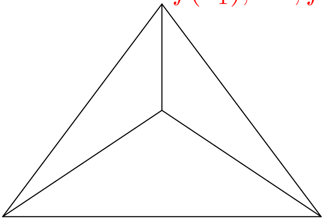

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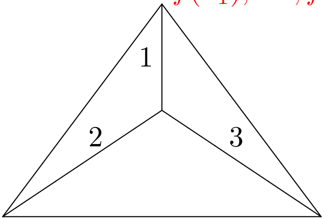

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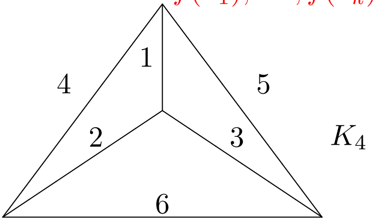

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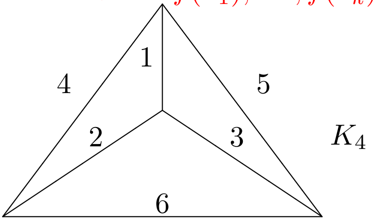
$$M = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$



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- query complexity $q(\epsilon)$ in terms of proximity parameter ϵ
  - this work: $q(\epsilon)$=big, super-exponential in $1/\epsilon$
  - [Bhattacharyya,Xie]: $q(\epsilon) = \omega(1/\epsilon)$
  - Is $q(\epsilon)$ at least super-polynomial in $1/\epsilon$?

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  - general case for V remains open