QUANTUM LOGARITHMIC SOBOLEV INEQUALITIES AND RAPID MIXING

Michael J. Kastoryano[1] and Kristan Temme[2]

[1] *Dahlem Center for Complex Quantum Systems, Freie Universität Berlin, 14195 Berlin, Germany*

[2] *Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

(Dated: October 5, 2012)

We introduce the tools of Log-Sobolev inequalities in the setting of quantum information theory [1]. The appropriate framework for thinking about Log-Sobolev inequalities is identified for quantum system, which allows us to rederive or generalize many of the classical results in the field [2]. Our work can be seen on the one hand as an extension/restriction to finite dimensional systems of [3], where non-commutative Log-Sobolev inequalities were first considered. On the other hand, it can be seen as a generalization to quantum systems of the semial work of Diaconis and Saloff-Coste [4], where Log-Sobolev inequalities were applied to analyze the mixing times of finite Markov chains. We furthermore resolve the notion of L_p-regularity, which is the main technical difficulty which separates the quantum from the classical theory, and we show that several important classes (unital and thermal) of quantum processes are well behaved in this special sense. Finally, we consider the mixing times of some specific examples, such as quantum expander maps.

Our analysis is restricted to finite dimentional state spaces. Therefore, by quantum Markov process, we simply mean a cpt map (quantum channel) with identical input and output space. We will restrict ourselves to cpt maps which form a one-parameter semigroup, and perform the analysis at the level of the semigroup. Furthermore, we restrict ourselves to primitive semigroups (i.e. ones with a unique full-rank stationary state) and to reversible semigroups (i.e. those whose generator is Hermitian with respect to the inner product $\langle f, g\rangle_\sigma \equiv \mathrm{tr}\left[\sqrt{\sigma} f^\dagger \sqrt{\sigma} g\right]$).

The ϵ-mixing time of a primitive semigroup is the minimum time after which one can certify that the output state is ϵ close to the stationary state in trace norm, starting from any initial state. If we denote σ be the stationary state of a semigroup generated by the quantum dynamical master equation $\partial_t \rho_t = \mathcal{L}(\rho_t)$, where $\mathcal{L}$ is the Liouvillian. Then, the mixing time is defined as

$$\tau_{mix}(\epsilon) = \min \{ t \,|\, \|\rho_t - \sigma\|_{tr} \leq \epsilon \text{ for all input states } \rho_0 \} \tag{1}$$

Hence, if one can find a systematic way of upper bounding $||\rho_t - \sigma||_1$ by a monotonically decreasing function in t, then we have a way of estimating the mixing time of the semigroup. It turns our that such a function can always be found and has the form

$$||\rho_t - \sigma||_1 \leq A e^{-bt} \tag{2}$$

In particular, if one chooses $b > \lambda$, where λ is the spectral gap of $\mathcal{L}$ - i.e. the smallest non-zero magnitude of the real part of the spectrum of $\mathcal{L}$ - then there always exists a t independent constant A satisfying the above bound. In [5], a systematic method was provided for choosing the constant A when $b = \lambda$ (note that this is not always possible if the semigroup is not reversible). One gets that for any state ρ,

$$||\rho_t - \sigma||_1^2 \leq \chi^2(\rho, \sigma) e^{-2t\lambda} \leq \frac{1}{\sigma_{\min}} e^{-2t\lambda}, \tag{3}$$

where $\chi^2(\rho,\sigma) = \mathrm{tr}\left[(\rho - \sigma)\Gamma_\sigma^{-1}(\rho - \sigma)\right]$ is the χ^2-divergence, and $\sigma_{\min}$ is the minimum eigenvalue of the stationary state σ. It is worth noting that $1/\sigma_{\min}$ is typically polynomial in the dimention of the system, and hence for lattice systems is exponential in the number of spins. This translates to a polynomial contribution of the prefactor to the mixing time.

The question which naturally arises is whether one can improve this bound in any systematic way. It turns out that Log-Sobolev inequalities provide exactly that. We show that the following holds

$$||\rho_t - \sigma||_1^2 \leq 2S(\rho||\sigma)e^{-2t\alpha} \leq 2\log[\frac{1}{\sigma_{\min}}]e^{-2t\alpha}, \tag{4}$$

where $S(\rho||\sigma)$ is the relative entropy, and α is a Log-Sobolev constant.

We can provide good estimates of $\sigma_{\min}$ for the two situations which are of particular interest to us: primitive unital semigroups, and thermal semigroups. For primitive unital semigroups of a d-dimensional system, $\sigma = \mathbb{1}/d$, and hence $1/\sigma_{\min} = d$. For thermal semigroups of an N-qubit system with Hamiltonian H at temperature β, the stationary state will be given by $\sigma_\beta = e^{-\beta H}/\mathrm{tr}\left[e^{-\beta H}\right]$. It is a straightforward calculation to see that we have the bound

$$\frac{1}{\sigma_{\min}} \leq de^{\beta(\lambda^{\max}(H)-\lambda^{\min}(H))} \tag{5}$$

Provided that the Hamiltonian is locally bounded and has only a polynomial number of terms, we get that for some positive constant $c \in \mathbb{R}$, $\sigma_{\min}^{-1} \leq e^{cN} = \mathcal{O}(ploy(d))$.

This implies that for both of our cases of interest, the pre-factor in the Log-Sobolev bound grows at most as $\log(d)$. Hence, its contribution the the mixing time is of the order of $\log(\log(d))$. This indicates that the Log-Sobolev constant gives a very strong estimate on the mixing time.

The majority of the work in [1] is devoted to characterising the constant α (in fact, a one-parameter family of Log-Sobolev constants is identified). We review here, without any technical detail, some of the results from [1].

Just as for the spectral gap, the Log-Sobolev constant is characterized variationally. Importantly, the variational form (Dirichlet form) for the spectral gap and for the Log-Sobolev constant are very similar, and hence one can sometimes mimic bounding methods on the spectral gap to obtain bounds on the Log-Sobolev constant. The natural language for the analysis of both χ^2 and Log-Sobolev inequalities is the so-called non-commutative $\mathbb{L}_p$ spaces [6]. We present these tools in a self contained way.

One of our main results is that, for reversible semigroups, $\alpha \leq \lambda$, hence the mixing time bounds are consistent. In applications, it is especially interesting when α and λ are not of the same order, because it indicates that the full spectrum of the channel contributes to the mixing time behavior. Another important result, is that the Log-Sobolev inequalities are essentially equivalent to the Hypercontractivity of the semigroup. Intuitively, this correspondance can be understood by the fact that the Log-Sobolev inequalities are an infinitesimal formulation of the global convergence behavior characterized by Hypercontractivity of the semigroup. As the Hypercontractive inequalities sweep through an entire family of operator norms, they provide a very complete characterization of contraction of the semigroup, which is why the mixing time bound based on the Log-Sobolev constant is often more accurate than the one based on the spectral gap alone. See [7] for a recent review of applications of Hypercontractivity to problems in quantum information theory.

As applications, we consider reversible unital channels, and thermal channels. Using the methods introduced in [1], we show that the Log Sobolev constant of the random unitary channel $T_{RU}(\rho) = \frac{1}{D}\sum_{j=1}^{D} U_j \rho U_j^\dagger$, where each U_j is chosen randomly according to the Haar measure can be bounded as

$$\frac{(1-2/d)\lambda}{\log(d-1)} \leq \alpha \leq \log D \frac{4+\log\log d}{2\log 3d/4} \tag{6}$$

Although T_{RU} is not a semigroup, using an elementary correspondence between channels and semigroups, we are able to make sense of the notion of a Log-Sobolev constant for a channel. It is important to note that the Log-Sobolev constant of the random unitary channel (quantum expander [8]) is upper and lower bounded by expressions which scale as $1/\log d$. Hence, this provides very strong evidence that the "randomness production" of quantum expanders cannot be faster that $\mathcal{O}(\log d)$ in general.

Finally, we mention a very appealing operational interpretation of the Log-Sobolev inequality of thermal semigroups[**?**]; i.e. semigroups which drive any inital state into the Gibbs state of some Hamiltonian H at temperature β. Then we can show that Log-Sobolev constant is simply the wheighted entropy production [10], or equivalently,

$$\alpha_1 = \inf_{\rho} \partial_t \log[F(\rho_t) - F(\rho_\beta)]|_{t=0}, \tag{7}$$

where $F(\rho) = \mathrm{tr}\,[\rho H] - \frac{1}{\beta}S(\rho)$ is the Free-Energy of state ρ, and ρ_β is the Gibbs staet of H. In other words, for thermal maps, the Log-Sobolev constant can be interpreted as the minimal normalized rate of change of the free energy of the system.

To conclude, we briefly discuss potential further application of the framework introduced in [1]. We have to point out, that we have to a large extent only discussed the formal setting of Log-Sobolev inequalities, and that many relevant applications remain to be worked out. In the classical setting, Log-Sobolev inequalities and hypercontractivity have been extremely useful tools. One area where they have proved to be paramount is in analyzing the mixing properties of spin systems on a lattice under Glauber dynamics. Several authors have been able to show a number of very tight mixing results [11], in particular relating spacial and temporal mixing in a one-to-one fashion. It would be very desirable to generalize these results to the quantum setting. More generally, a number of methods, including block renormalization transformations and comparison theorems, have been developed in the classical setting in order to explicitly calculate the Log-Sobolev constant for specific systems.

Most importantly perhaps, the Log-Sobolev constant (and not the spectral gap) of a Liouvillian $\mathcal{L}$ is the natural open systems analogue of the gap of the Hamiltonian for closed systems for many statements in the many body theory of lattice systems. For instance, a systems size independant Log-Sobolev constant of a local Liouvillian is expected to imply i) clustering of correlations in the stationary state, ii) and Area Law (with logarithmic corrections), and iii) stability of stationary states [12].

---

[1] M. J. Kastoryano and K. Temme, Quantum logarithmic Sobolev inequalities and rapid mixing *Arxiv preprint arXiv:1207.3261*, 2012.

[2] L. Gross. Logarithmic sobolev inequalities and contractivity properties of semigroups. *Lecture Notes in Mathematics*, 1563:54–88, 1993.

[3] Olkiewicz R. and Zegarlinski B. Hypercontractivity in noncommutative $\mathbf{L}_p$ spaces. *Journal of Functional Analysis*, 161(1):246–285, 1999.

[4] P. Diaconis and L. Saloff-Coste. Logarithmic sobolev inequalities for finite markov chains. *The Annals of Applied Probability*, 6(3):695–750, 1996.

[5] K. Temme, MJ Kastoryano, MB Ruskai, MM Wolf, and F. Verstraete. The χ2-divergence and mixing times of quantum markov processes. *Journal of Mathematical Physics*, 51(12):122201–122201, 2010.

[6] U. Haagerup. Lp-spaces associated with an arbitrary von neumann algebra. *Algebres dopérateurs et leurs applications en physique mathématique, CNRS*, 15:175–184, 1979.

[7] A. Montanaro Some applications of hypercontractive inequalities in quantum information theory. arXiv:1208.0161

[8] M. B. Hastings. Random unitaries give quantum expanders. *Phys. Rev. A*, 76:032315, 2007.

[9] E.B. Davies. Generators of dynamical semigroups. *Journal of Functional Analysis*, 34(3):421–432, 1979.

[10] H. Spohn. Entropy production for quantum dynamical semigroups. *Journal of Mathematical Physics*, 19:1227, 1978.

[11] F. Martinelli. Lectures on glauber dynamics for discrete spin models. *Lectures on probability theory and statistics (Saint-Flour, 1997)*, 1717:93–191, 1997.

[12] M. J. Kastoryano et. al. Correlations and Areal Laws for local open systems.