

Quantum logarithmic Sobolev inequalities and rapid mixing

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Outline

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 - Setting
 - Convergence rates
- 2 Results
 - Mixing times
 - Mathematical results
- 3 Applications and outlook
 - Quantum Expanders
 - Liouvillian Complexity

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- If $\Gamma_\sigma \mathcal{L} = \mathcal{L}^* \Gamma_\sigma$, where σ is the stationary state of \mathcal{L} and $\Gamma_\sigma(X) = \sqrt{\sigma} X \sqrt{\sigma}$, the \mathcal{L} is *reversible*.

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Note: we do not yet make any assumptions about locality or geometry at this point.

The question

Let \mathcal{L} be the generator of a primitive reversible quantum dynamical semigroup. Given $\epsilon > 0$, for what $\tau \geq t > 0$ do we have

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The answer: general convergence theorem

Let $\lambda > 0$ be the *spectral gap* of \mathcal{L} , then for any $b \leq \lambda$, there exists a finite A such that

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What are good choices for A and b ? We will argue that the Log Sobolev machinery is the finest available to answer this question.

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- 3 Liouvillian complexity: what can we say about systems whose Log Sobolev constant is independent of the system size?
- 4 Dissipative algorithms?
- 5 Concentration of measure?

A few definitions to start with...

non-commutative \mathbb{L}_p spaces

- The \mathbb{L}_p inner product. For two hermitian operators f, g :

$$\langle f, g \rangle_\sigma = \text{tr}[\Gamma_\sigma(f)g] \equiv \text{tr}[\sigma^{1/2}f\sigma^{1/2}g]. \quad (4)$$

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- The \mathbb{L}_p norm. For any hermitian operator f :

$$\|f\|_{p,\sigma} = \text{tr}[|\Gamma_\sigma^{1/p}(f)|^p]^{1/p} \quad (5)$$

A few more definitions...

Variance and Entropy functionals

- The variance

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- The \mathbb{L}_p relative entropies. For any hermitian operator f :

$$\text{Ent}_1(f) = \text{tr}[\Gamma_\sigma(f)(\log(\Gamma_\sigma(f)) - \log(\sigma))] \quad (7)$$

$$- \text{tr}[\Gamma_\sigma(f)] \log(\text{tr}[\Gamma_\sigma(f)]) \quad (8)$$

$$\text{Ent}_2(f) = \text{tr}\left[\left(\Gamma_\sigma^{1/2}(f)\right)^2 \log\left(\Gamma_\sigma^{1/2}(f)\right)\right] \quad (9)$$

$$- \frac{1}{2} \text{tr}\left[\left(\Gamma_\sigma^{1/2}(f)\right)^2 \log(\sigma)\right]$$

$$- \frac{1}{2} \|f\|_{2,\sigma}^2 \log(\|f\|_{2,\sigma}^2).$$

Yet more... (sorry!)

Dirichlet Forms

$$\mathcal{E}_1(f) = -\frac{1}{2} \text{tr}[\Gamma_\sigma(\mathcal{L}(f))(\log(\Gamma_\sigma(f)) - \log(\sigma))] \quad (10)$$

$$\mathcal{E}_2(f) = -\langle f, \mathcal{L}(f) \rangle_\sigma. \quad (11)$$

Useful identities:

$$\text{Var}(\Gamma_\sigma^{-1}(\rho)) = \chi^2(\rho, \sigma), \quad \text{Ent}_2(\Gamma_\sigma^{-1}(\rho)) = S(\rho||\sigma) \quad (12)$$

Spectral Gap and Log-Sobolev constant

- The spectral gap of \mathcal{L} :

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Note: one can in fact define a whole family of Log Sobolev constants α_p , with $p \geq 0$.

Theorem

Let \mathcal{L} denote the generator of a primitive reversible semigroup with fixed point σ . Then,

① χ^2 bound:

$$\begin{aligned} \|\rho_t - \sigma\|_1 &\leq \sqrt{\chi^2(\rho_t, \sigma)} & (15) \\ &\leq \sqrt{\chi^2(\rho, \sigma)} e^{-\lambda t} \leq \sqrt{1/\sigma_{\min}} e^{-\lambda t}. \end{aligned}$$

Where σ_{\min} denotes the smallest eigenvalue of the fixed point σ .

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❷ Log-Sobolev bound:

$$\begin{aligned} \|\rho_t - \sigma\|_1 &\leq \sqrt{2S(\rho_t || \sigma)} & (16) \\ &\leq \sqrt{2S(\rho || \sigma)} e^{-\alpha_1 t} \leq \sqrt{2 \log(1/\sigma_{\min})} e^{-\alpha_1 t}. \end{aligned}$$

Where σ_{\min} denotes the smallest eigenvalue of the fixed point σ .

Interpretation

Davies generators describe the dissipative dynamics resulting as the weak (or singular) coupling limit of a system coupled to a large heat bath. For these *thermal maps*, the Log-Sobolev constant is the minimal normalized rate of change of the free energy of the system:

$$\alpha_1 = \inf_{\rho} \partial_t \log [F(\rho_t) - F(\rho_\beta)]|_{t=0}, \quad (17)$$

where $F(\rho) = \text{tr}[\rho H] - \frac{1}{\beta} S(\rho)$ is the free energy of the system, and ρ_β is the Gibbs state.



Mathematical results

Theorem (Partial ordering)

Let \mathcal{L} be a primitive reversible Liouvillian with stationary state σ . The Log-Sobolev constants α_1 , α_2 and the spectral gap λ of \mathcal{L} are related as:

$$\alpha_2 \leq \alpha_1 \leq \lambda. \quad (18)$$

Theorem (Hypercontractivity)

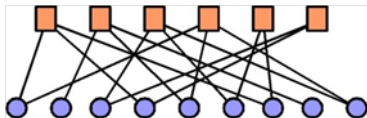
Let \mathcal{L} be a primitive Liouvillian with stationary state σ , and let $T_t = e^{t\mathcal{L}}$ be its associated semigroup. Then

- 1 If there exists a $\alpha > 0$ such that $\|T_t\|_{(2,\sigma) \rightarrow (p(t),\sigma)} \leq 1$ for all $t > 0$ and $2 \leq p(t) \leq 1 + e^{2\alpha t}$. Then \mathcal{L} satisfies LS_2 with $\alpha_2 \geq \alpha$.
- 2 If \mathcal{L} is weakly \mathbb{L}_p -regular, and has an LS_2 constant α_2 , then $\|T_t\|_{(2,\sigma) \rightarrow (p(t),\sigma)} \leq 1$ for all $t > 0$ when $2 \leq p(t) \leq 1 + e^{2\alpha_2 t}$. If, furthermore, \mathcal{L} is strongly \mathbb{L}_p regular, then the above holds for all $t > 0$ when $2 \leq p(t) \leq 1 + e^{4\alpha_2 t}$.

Quantum expanders

Quantum Expander: (sequence of) quantum channel with i) a fixed number of Kraus operators (D), and ii) the spectral gap λ of the channel is asymptotically independent of dimension d . Then,

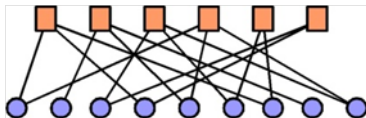
$$\frac{(1 - 2/d)\lambda}{\log(d - 1)} \leq \alpha_2 \leq \log D \frac{4 + \log \log d}{2 \log 3d/4} \quad (19)$$



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The mixing time is of order $\log d$

Suppose that \mathcal{L} describes the open system dynamics on a lattice of qudits. Assume furthermore that \mathcal{L} is: i) **primitive and reversible**, ii) **local**, and iii) has a Log Sobolev constant α_1 which is **system size independent**. Then we get

(strong) clustering of correlations

$$\langle O_A O_B \rangle_\sigma - \langle O_A \rangle_\sigma \langle O_B \rangle_\sigma \leq K \log \left(\frac{1}{\sigma_{\min}} \right) e^{-\alpha_1 d(A,B)/v} \quad (20)$$

where K is volume like.

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Stability of Liouvillians

Let \mathcal{Q} be a local perturbation of \mathcal{L} , and $\mathcal{L}' = \mathcal{L} + \mathcal{Q}$ with stationary state σ' , then

$$\|\sigma - \sigma'\|_1 \leq \frac{\|\mathcal{Q}\|_{1-1}}{\alpha_1} \left(\log \left(\log \left(\frac{1}{\sigma_{\min}} \right) \right) + 1 \right) \quad (21)$$



Thank you for your attention!

References



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