Recoupling Coefficients and Quantum Entropies arXiv:1210.0463

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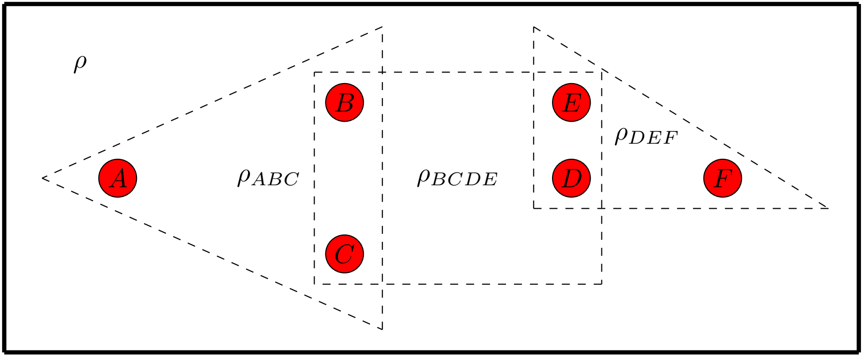
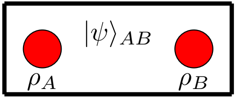
joint work with Matthias Christandl and Michael Walter

Quantum Marginal Problem (QMP)

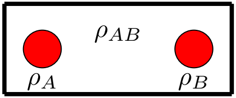


Is there a global quantum state ρ which is compatible with given reduced density matrices $\rho_{ABC}, \rho_{BCDE}, \rho_{DEF}, \ldots$?

History

1. 

$\Leftrightarrow r_A = r_B$. $(|\psi_{AB}\rangle = \sum_i \sqrt{r_i}|i\rangle_A|i\rangle_B.)$

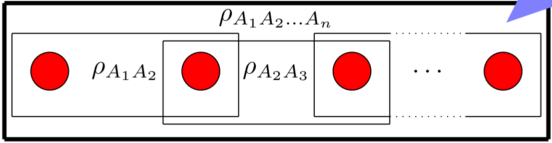
2. Klyachko(2004), Christandl & Mitchison(2004) & Harrow(2005) :


$\Leftrightarrow g_{\lambda,\alpha,\beta} \neq 0$ for $\lambda \sim r_{AB}, \alpha \sim r_A, \beta \sim r_B$.

   Klyachko(2004): Inequalities between eigenvalues

3. Liu(2006):

   Overlapping subsystems!



: QMA-complete.

von Neumann Entropy Inequalities

$$H(\rho) := -\operatorname{tr} \rho \log \rho = -\sum_i r_i \log r_i$$

Eigenvalue spectra $\Rightarrow$ von Neumann entropy

Von Neumann entropy $\rightsquigarrow$ Eigenvalue spectra

History of inequalities

1. $n = 2$: $H(A) + H(B) \geq H(AB)$ - Subadditivity
2. $n = 3$: $H(AB) + H(BC) - H(B) - H(ABC) \geq 0$ - Strong Subaddivity (Lieb & Ruskai- 1973)
3. $n = 4$: More inequalities?
   Infinitely many constrained inequalities (Cadney, Linden and Winter - 2011)
4. For Shannon entropies: Infinitely many independent inequalities (Zhang & Yeung - 1998, Matus - 2007)

Young diagrams

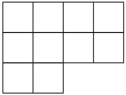
A Young diagram λ, of k boxes and at most d rows, is a d-tuple of nonnegative integers:

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$$

such that

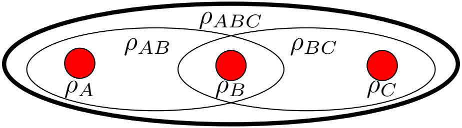
$$\lambda_i \geq \lambda_{i+1} \geq 0, \quad \sum_i \lambda_i = k.$$

Example: $\lambda = (4, 4, 2)$



A normalized version of λ is $\overline{\lambda} := (\lambda_1/k, \lambda_2/k, \ldots, \lambda_d/k)$:
$\lambda = (4, 4, 2) \Rightarrow \overline{\lambda} = (0.4, 0.4, 0.2)$.

$\sim$ eigenvalue spectra



Theorem

$\exists$ ρ_{ABC} *with spectra* $r_A, r_B, r_C, r_{AB}, r_{BC}, r_{ABC}$

iff

$\exists$ *sequence of Young diagrams,* $\alpha, \beta, \gamma, \mu, \nu, \lambda$ *with* k *boxes st*

$$(r_A, r_B, r_C, r_{AB}, r_{BC}, r_{ABC}) = \lim_{k\to\infty} \left(\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\mu}, \overline{\nu}, \overline{\lambda}\right) \textit{ and}$$

$$\left\| \begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix} \right\| \geq \frac{1}{\mathrm{poly}(k)}$$

Recoupling coefficients of S_k

Corollary

Symmetry of Recoupling Coefficients $\Rightarrow$ **Strong Subadditivity of von Neumann entropy**

Representations of Symmetric group

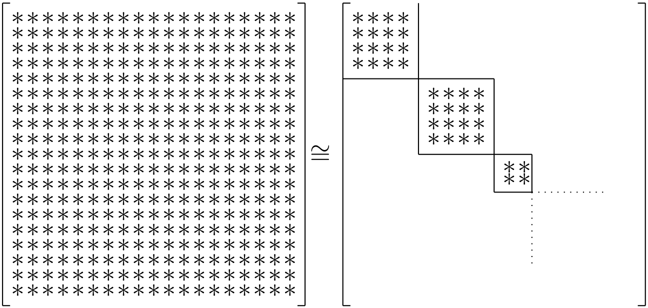
Let V be a vector space on which S_k acts. It can be decomposed into irreducible representations V_λ:

$$V \cong \bigoplus_\lambda V_\lambda \otimes H_\lambda^V$$

λ: Young diagrams

multiplicity space

$S_k \ni \pi \to$



Recoupling coefficients

Clebsch-Gordan isomorphism for S_k:

$$V_\alpha \otimes V_\beta \cong \bigoplus_\lambda V_\lambda \otimes H_\lambda^{\alpha\beta}$$

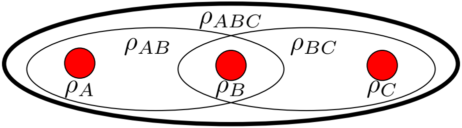
Clebsch-Gordan in two different ways:

$$\begin{aligned} V_\alpha \otimes V_\beta \otimes V_\gamma &\cong \bigoplus_{\lambda,\mu} V_\lambda \otimes H_\mu^{\alpha\beta} \otimes H_\lambda^{\mu\gamma} \\ &\cong \bigoplus_{\lambda,\nu} V_\lambda \otimes H_{\beta\gamma}^{\nu} \otimes H_{\alpha\nu}^{\lambda} \end{aligned}$$

$$\begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix} : H_\mu^{\alpha\beta} \otimes H_\lambda^{\mu\gamma} \to H_{\beta\gamma}^{\nu} \otimes H_{\alpha\nu}^{\lambda}$$

Recoupling!

Recoupling coefficients and Tripartite spectra



Theorem

$\exists\ \rho_{ABC}$ *with spectra* $r_A, r_B, r_C, r_{AB}, r_{BC}, r_{ABC}$

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$$(r_A, r_B, r_C, r_{AB}, r_{BC}, r_{ABC}) = \lim_{k\to\infty} \left(\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\mu}, \overline{\nu}, \overline{\lambda}\right) \text{ and}$$

$$\left\| \begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix} \right\| \geq \frac{1}{\mathrm{poly}(k)}$$

$$H_{\mu}^{\alpha\beta} \otimes H_{\lambda}^{\mu\gamma} \to H_{\beta\gamma}^{\nu} \otimes H_{\alpha\nu}^{\lambda}$$

Proof-1: Spectrum estimation

S_k permutes the k-copies of $\mathbb{C}^d$,

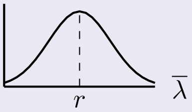
Schur-Weyl duality

$$(\mathbb{C}^d)^{\otimes k} \cong \bigoplus_{\lambda\text{:k boxes,d rows}} \underbrace{V_\lambda \otimes \mathcal{U}_\lambda^d}_{\uparrow P_\lambda \uparrow}$$

Theorem (Keyl & Werner - 2001)

Let ρ be a quantum state with eigenvalue spectra r, then

$$\mathrm{tr}(P_\lambda \rho^{\otimes k}) \leq \exp(-k\|\overline{\lambda} - r\|_1^2/2) \quad \rightarrow \quad \mathrm{tr}(P_\lambda \rho^{\otimes k}) \leq$$



$$P^\delta := \sum_{\lambda:\|\overline{\lambda}-r\|_1 \leq \delta} P_\lambda \Rightarrow \quad \mathrm{tr}(P^\delta \rho^{\otimes k}) \geq 1 - \exp(-k\delta^2/2) \approx 1$$

Proof-2: Recouplings from tripartite Schur-Weyl

$$\left(\mathbb{C}^{abc} \cong \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c\right)^{\otimes k} \cong \begin{cases} \bigoplus V_\lambda \otimes \mathrm{H}_\lambda^{\mu\gamma} \otimes \mathrm{H}_\mu^{\alpha\beta} \otimes \mathcal{U}_\alpha^a \otimes \mathcal{U}_\beta^b \otimes \mathcal{U}_\gamma^c \\ \uparrow Q \uparrow \\ \bigoplus \overbrace{\underbrace{V_\alpha \otimes V_\beta \otimes V_\gamma \otimes \mathcal{U}_\alpha^a \otimes \mathcal{U}_\beta^b \otimes \mathcal{U}_\gamma^c}}\\ \downarrow P \downarrow \\ \bigoplus V_\lambda \otimes \mathrm{H}_\lambda^{\alpha\nu} \otimes \mathrm{H}_\nu^{\beta\gamma} \otimes \mathcal{U}_\alpha^a \otimes \mathcal{U}_\beta^b \otimes \mathcal{U}_\gamma^c \end{cases}$$

$$P.Q = \mathbf{1}_{V_\lambda \otimes \mathcal{U}_\alpha^a \otimes \mathcal{U}_\beta^b \otimes \mathcal{U}_\gamma^c} \otimes \begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix}$$

Proof-3: Golden shot

Take a ρ_{ABC} with spectra $r_A, r_B, r_C, r_{AB}, r_{BC}, r_{ABC}$

$$\Downarrow$$

$$|\operatorname{tr}(P^\delta Q^\delta \rho_{ABC}^{\otimes k})| \geq 1 - \exp(-k\delta^2/2)$$

$$\Downarrow$$

$$\|P^\delta Q^\delta\|_\infty \geq 1 - \exp(-k\delta^2/2)$$

Use $P.Q = \mathbf{1}_{V_\lambda \otimes \mathcal{U}^a_\alpha \otimes \mathcal{U}^b_\beta \otimes \mathcal{U}^c_\gamma} \otimes \begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix}$

$$\sum_{\alpha,\beta,\gamma,\mu,\nu,\lambda} \| \begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix} \| \geq 1 - \exp(-k\delta^2/2)$$

$$\Downarrow$$

$$\| \begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix} \| \geq \frac{1}{\operatorname{poly}(k)} : \quad \overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\mu}, \overline{\nu}, \overline{\lambda} \sim r_A, r_B, r_C, r_{AC}, r_{BC}, r_{ABC}$$

□

Corollary

Symmetry of Recoupling Coefficients $\Rightarrow$ **Strong Subadditivity of von Neumann entropy**

Proof ingredients:

1. $$\left\|\begin{bmatrix}\alpha & \beta & \mu \\ \gamma & \lambda & \nu\end{bmatrix}\right\|^2 = \frac{\dim V_\mu \dim V_\nu}{\dim V_\beta \dim V_\lambda}\left\|\begin{bmatrix}\alpha & \mu & \beta \\ \gamma & \nu & \lambda\end{bmatrix}\right\|^2,$$ Symmetry!
2. Theorem
3. $$\dim V_\lambda \asymp \frac{k!}{\lambda_1!\lambda_2!\ldots\lambda_d!} \asymp 2^{kH(\overline{\lambda})}$$ Stirling's formula

Proof of Corollary

1. $$\left\| \begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix} \right\|^2 = \frac{\dim V_\mu \dim V_\nu}{\dim V_\beta \dim V_\lambda} \left\| \begin{bmatrix} \alpha & \mu & \beta \\ \gamma & \nu & \lambda \end{bmatrix} \right\|^2$$

   Proof via graphical calculus!

2. Theorem implies

   $$\frac{\dim V_\mu \dim V_\nu}{\dim V_\beta \dim V_\lambda} \geq \frac{1}{\mathrm{poly}(k)} \quad \text{for} \quad \begin{matrix} \overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\mu}, \overline{\nu}, \overline{\lambda} \\ \sim \\ r_A, r_B, r_C, r_{AB}, r_{BC}, r_{ABC} \end{matrix}$$

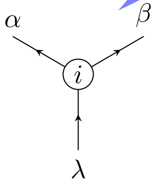
3. Use $\lim_{k\to\infty} \dfrac{1}{k} \log \dim V_\lambda = H(\overline{\lambda})$:

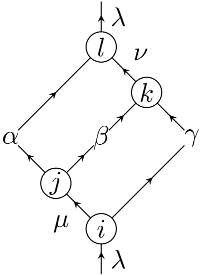
   $$H(\rho_{AB}) + H(\rho_{BC}) - H(\rho_B) - H(\rho_{ABC}) \geq 0 \qquad \square$$

Proof of Symmetry: Graphical calculus

S_k invariant maps

Preskill's notes: topological q.c.

$H_\lambda^{\alpha\beta} \ni |i\rangle =$ 

$\Rightarrow \mathbf{1}_{V_\lambda} \otimes \begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix}_{ij}^{kl} =$ 

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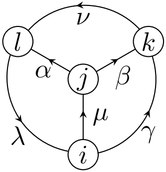
Trace over V_λ and deform,

$$\begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix}_{ij}^{kl} = \frac{1}{\dim V_\lambda}$$ 

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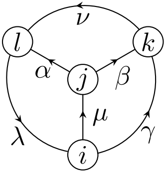
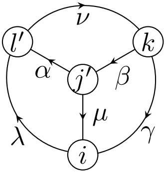
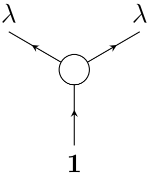
Proof of Symmetry

Prove $\begin{bmatrix}\alpha & \beta & \mu \\ \gamma & \lambda & \nu\end{bmatrix}_{ij}^{kl} = \frac{\sqrt{\dim V_\mu \dim V_\nu}}{\sqrt{\dim V_\beta \dim V_\lambda}} \sum_{j'l'} \overline{U}_{jj'} V_{ll'} \begin{bmatrix}\alpha & \mu & \beta \\ \gamma & \nu & \lambda\end{bmatrix}_{ij'}^{kl'}$

$\frac{1}{\dim V_\lambda}$ 

$= \frac{\sqrt{\dim V_\mu \dim V_\nu}}{\sqrt{\dim V_\beta \dim V_\lambda}} \sum_{j'l'} \frac{\overline{U}_{jj'} V_{ll'}}{\dim V_\nu}$ 

Use the self duality of V_λ:

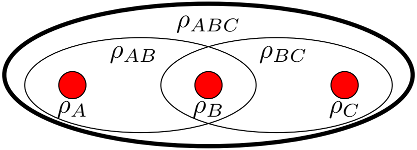
$:=$ 

$= \frac{1}{\sqrt{\dim V_\lambda}} \sum |e_\lambda\rangle |e_\lambda\rangle$ □

Conclusions

Asymptotic of S_k recoupling coefficients

$\sim$



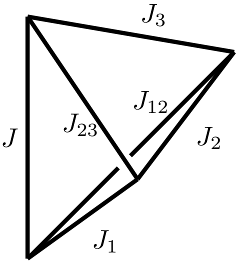
Symmetry of Recoupling

$\Downarrow$

Strong subadditivity

Asymptotic of $\mathrm{SU}(2)$ (Wigner) $6j$-symbols

$\sim$



- Spectrum of ρ_{AC} is missing!
- Symmetry of other objects $\Rightarrow$ other entropy inequalities ?