Negative Quasi-Probability as a Resource for Quantum Computation

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Big Picture

Big Picture Question

What resources are necessary and sufficient for quantum computational speedup?

Resources for Quantum Computation?

Some Candidates

- Entanglement?
- Purity?
- Coherence?
- Discord? (probably not)

Quantum Resources

Resources arise from operational restrictions on the quantum formalism.

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Resources for Fault Tolerance

Goal

The goal is to characterize resources for fault tolerant quantum computation.

Fault Tolerance

- Stabilizer operations are a typical fault tolerant set.
- This defines a natural restriction on the set of quantum operations.
- This set is efficiently simulatable by the Gottesman-Knill protocol.
- Thus we need injection of resource states.

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Magic State Computing (Bravyi, Kitaev 2005)

Magic State Model

- Operational restriction: perfect stabilizer operations (states, gates and projective measurement)
- Additional resource: preparation of non-stabilizer state ρ_R

Magic State Distillation

- Consume many resource states ρ_R to produce a few very pure resource states $\tilde{\rho}_R$
- Inject $\tilde{\rho}_R$ to perform non-stabilizer unitary gates (using only fault tolerant stabilizer operations)

A Sharper Question

Do all non-stabilizer states promote stabilizer computation to quantum computation?

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Main Result: Bound Magic States for Odd Dimension

- There is a large class of non-stabilizer quantum states (*bound magic states)* that are not useful for magic state distillation.
- Quantum circuits composed of stabilizer operations composed of stabilizer operations and bound magic states are efficiently classically simulatable. This is an extension of Gottesman-Knill to non-stabilizer inputs.

Quasi-Probability Representation

Quasi-Prob. Representation

A linear map from Hermitian operators to real numbers, $W : L(\mathscr{H}_{d^n}) \to \mathbb{R}^{d^{2n}}$. In particular:

- quantum states $\to$ quasi-probability distributions
- POVM elements $\to$ conditional quasi-probability distributions.

Negativity

Some states/measurements must be *negatively represented.* (Emerson, Ferrie 2009).

Subtheory

Choice of positively represented subtheory is largely arbitrary.

Discrete Wigner Representation for Odd Dimension

Insight

Choice of quasi-probability representation can reflect operational restriction.

Wigner Representation

Stabilizer operations have positive representation. (Gross 2006)

Negative Probabilities

Ancilla preparation may be negatively represented.

1/3	1/3	1/3
0	0	0
0	0	0

Figure: Wigner representation of qutrit $|0\rangle$ state

1/6	1/6	1/6
1/6	-1/3	1/6
1/6	1/6	1/6

Figure: Wigner representation of qutrit $|0\rangle - |1\rangle$ state

Stabilizer Operations Preserve Positive Representation

Observation

Negative Wigner representation is a resource that can not be created by stabilizer operations.

Proof

Let $\rho \in L(\mathbb{C}_{d^n})$ be an n qudit quantum state with positive Wigner representation. Observe the following:

1. $U\rho U^\dagger$ is positively represented for any Clifford (stabilizer) unitary U.
2. $\rho \otimes S$ is positively represented for any stabilizer state S.
3. $M\rho M / \mathrm{Tr}(M\rho M)$ is positively represented for any stabilizer projector M.

So What?

Discrete Hudson's Theorem (Gross 2006)

Pure states have positive representation if and only if they are stabilizer states.

Positive Representation $\equiv$ Stabilizer State?

Do all non-stabilizer states have negative Wigner representation?

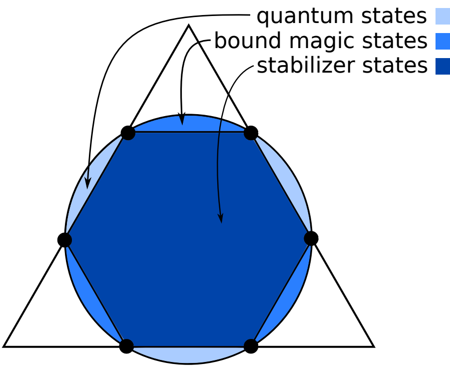
Stabilizer Polytope

Stabilizer Polytope

- Define convex polytope with stabilizer states as its vertices
- Can be equivalently defined by set of halfspaces - “facets”

Non-Negativity Specifies Facets

The Wigner simplex has d^2 facets, shared with the stabilizer polytope



Slice of the Stabilizer Polytope

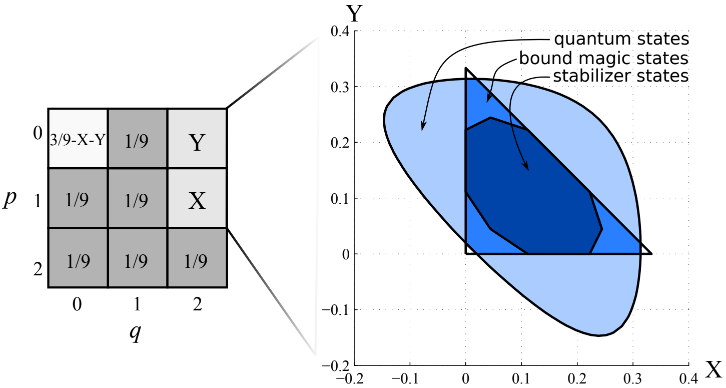


Figure: Slice defined by fixing six entries of the Wigner function and varying the remaining through their possible values to create the plot.

Extended Gottesman-Knill Theorem

Scope

- Prepare ρ with positive representation
- Act on input with Clifford U_F (corresponding to linear size F)
- Perform measurement $\{E_k\}$ with positive representation

Simulation Protocol

- Sample phase space point (u, v) according to distribution $W_\rho(u, v)$
- Evolve phase space point according to $(u, v) \to \boldsymbol{F}^{-1}(u, v)$
- Sample from measurement outcome according to $\tilde{W}_{\{E_k\}}(u, v)$

See also *Positive Wigner functions render classical simulation of quantum computation efficient*, A. Mari and J. Eisert

Linear Optics

Odd Dimension	*Infinite Dimension*
Stabilizer Operations	Linear Optics
Stabilizer States	Gaussian States
Discrete Wigner Function	Wigner Function

Table: Comparison of Odd and Infinite Dimensional Formalisms

Results

- There exist mixed states with positive Wigner representation that are not convex combinations of gaussian states (Bröcker and Werner 1995)
- Computations using linear optical transformations and measurements as well as preparations with positive Wigner function can be efficiently classically simulated.[a]

[a] Veitch, Wiebe, Ferrie and Emerson (2012)

Summary and Open Problems

Summary

- Negative Wigner representation resource for stabilizer restriction
- Extended Gottesman-Knill
- Bound states for magic state distillation

Future Work

- Does this extend to other operational restrictions?
- Is negativity sufficient for distillability?
- Resource theory for stabilizer formalism?

Paper Reference

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