Spectral convergence bounds for classical and quantum Markov processes

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Classical and quantum Markov chains

Markov chain: Description of time-homogenous probabilistic evolution.

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$$\begin{array}{rlrl} \mathcal{X}: & \text{state space,} & \rho: & \text{state of system,} \\ \mathcal{T}: & \text{transition map,} & \mathcal{T}_\infty: & \text{asymptotic evolution} \end{array}$$

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Classical:

- $\mathcal{X} = \mathbb{R}^d$
- ρ: vector with non-negative components, sum to 1
- $\mathcal{T}$: stochastic matrix

Quantum:

- $\mathcal{X} = \{X \in \mathbb{C}^{d\times d} | X = X^\dagger\}$
- ρ: positive semi-definite trace-one matrix
- $\mathcal{T}$: trace-preserving and completely positive map

Approaching Asymptotic behavior

In many cases one is interested, when asymptotic behavior sets in:

Classical:

- Algorithms close to correct?
- Shuffling random?

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- Dissipative state preparation and computation

- Stability of fixed point of evolution
- Cut-off phenomena

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How is the *spectrum* of $\mathcal{T}$ related to $\|\mathcal{T}^n - \mathcal{T}_\infty^n\|$?

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Mathematical primer

Linear maps $\mathcal{M}$:

- $\sigma(\mathcal{M}) = \{\lambda_1, ..., \lambda_d\}$ spectrum of $\mathcal{M}$ with spectral radius $\mu_{\mathcal{M}}$,
- $m_{\mathcal{M}}(z) = \prod_i (z - \lambda_i)^{k_i}$ minimal polynomial of $\mathcal{M}$: smallest degree non-zero poly. with $m_{\mathcal{M}}(\mathcal{M}) = 0$

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- Define

$$\mathcal{T}_\infty := \sum_{|\lambda_i|=1} \lambda_i \mathcal{P}_i$$

via Jordan decomposition: $\mathcal{T} = \sum_i (\lambda_i \mathcal{P}_i + \mathcal{N}_i)$, $\mathcal{P}_i$ spectral projector, $\mathcal{N}_i$ nilpotent.

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- $\mathcal{T}^n - \mathcal{T}_\infty^n = (\mathcal{T} - \mathcal{T}_\infty)^n$

Linear algebraic bounds

Use $\|\mathcal{T}^n - \mathcal{T}_\infty^n\| = \|(\mathcal{T} - \mathcal{T}_\infty)^n\|$ and Jordan/ Schur decompositions of $\mathcal{T} - \mathcal{T}_\infty$.

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Jordan:

Let $\mu = \mu_{\mathcal{T}-\mathcal{T}_\infty}$ and d_μ largest Jordan block for μ. There are n-independent $C_1, C_2 > 0$ such that

$$C_1 \mu^{n-d_\mu+1} n^{d_\mu-1} \leq \|\mathcal{T}^n - \mathcal{T}_\infty^n\| \leq C_2 \mu^{n-d_\mu+1} n^{d_\mu-1},$$

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Schur: (for quantum channels)

$$\|\mathcal{T}^n - \mathcal{T}_\infty^n\|_\diamond \leq 2d^{3/2}(\mu + 2d^{1/2})^{d^2-1} n^{d^2-1} \mu^{n-d^2+1}.$$

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Both bounds are not satisfactory: Jordan only qualitative, Schur too bad.

Mathematical primer II

Certain spaces of analytic functions:

- $Hol(\mathbb{D})$: space of analytic functions on complex unit disc.
- $H^p \subset Hol(\mathbb{D})$ with $p > 0$: Hardy spaces

$$H^p = \{f \in Hol(\mathbb{D}) | \, \|f\|_{H^p}^p := \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\phi})|^p \mathrm{d}\phi < \infty\}$$

- $W \subset Hol(\mathbb{D})$: Wiener algebra of absolutely convergent Taylor series

$$W = \{f = \sum_{k \geq 0} \hat{f}(k) z^k | \sum_{k \geq 0} |\hat{f}(k)| < \infty\}.$$

Power-bounded operators obey Wiener functional calculus

$\mathcal{M}$ power-bounded iff $\|\mathcal{M}^n\| \leq C \ \forall n \in \mathbb{N}$. Examples:

- $\mathcal{T}$ quantum channel: $\|\mathcal{T}^n\|_\diamond = 1$
- $\mathcal{T}$ classical stochastic matrix: $\|\mathcal{T}^n\|_{1\to 1} = 1$
- $\mathcal{T} - \mathcal{T}_\infty$: $\|(\mathcal{T} - \mathcal{T}_\infty)^n\|_\diamond = \|\mathcal{T}^n - \mathcal{T}_\infty^n\|_\diamond \leq \|\mathcal{T}^n\|_\diamond + \|\mathcal{T}_\infty^n\|_\diamond = 2$

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Observation I:

$$\|f(\mathcal{M})\| = \|\sum_{k\geq 0} \hat{f}(k)\mathcal{M}^k\| \leq \sum_{k\geq 0} |\hat{f}(k)| \, \|\mathcal{M}^k\| \leq C \sum_{k\geq 0} |\hat{f}(k)| = C\,\|f\|_W$$

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Observation II:

$$\|f(\mathcal{M})\| = \|(f + m_\mathcal{M} g)(\mathcal{M})\| \leq C \, \|f + m_\mathcal{M} g\|_W \;\; \forall g \in W$$

Bounding functions of operators

Thus, $\|f(\mathcal{M})\| \leq C \inf_{g \in W} \|f + m_{\mathcal{M}} g\|_W$
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- Find "good" function space for given class of operators
- Use above to shift problem to function space
- Find bound in function space e.g choose "good" h with $\inf_{g \in S} \|f + m_{\mathcal{M}} g\|_S \leq \|h\|_S$

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Examples:

- $\mathcal{M}$ Hilbert space contraction, then $\|f(\mathcal{M})\| \leq \inf_{g\in H^\infty} \|f + m_{\mathcal{M}} g\|_{H^\infty} \;\; \forall f \in H^\infty$
- $\mathcal{T}$ quantum channel, then [Nik06] $\left\|\mathcal{T}^{-1}\right\|_\diamond \leq \sqrt{2e}d/(\prod_i |\lambda_i|)$
- $\mathcal{T}$ quantum channel, then

$$\|\mathcal{T}^n - \mathcal{T}_\infty^n\|_\diamond = \|(\mathcal{T} - \mathcal{T}_\infty)^n\|_\diamond \leq 2 \inf_{g\in W} \left\|z^n + g\, m_{(\mathcal{T}-\mathcal{T}_\infty)}\right\|_W$$

Main result: Spectrum and convergence

Theorem (Szehr, Reeb, Wolf [SRW13])

Suppose $\|\mathcal{T}^n\| \leq C \ \forall n \in \mathbb{N}$. Let $m = m_{\mathcal{T}-\mathcal{T}_\infty}$ be minimal polynomial and μ spectral radius of $\mathcal{T} - \mathcal{T}_\infty$. Then, for $n > \frac{\mu}{1-\mu}$ we have

$$\|\mathcal{T}^n - \mathcal{T}_\infty^n\| \leq \mu^n \, R(\mu, m, n) \prod_{m/(z-\lambda_D)} \frac{1-(1+\frac{1}{n})\mu|\lambda_i|}{\mu - |\lambda_i| + \frac{\mu}{n}},$$

where $R(\mu, m, n) = \frac{4Ce^2\sqrt{|m|}(|m|+1)}{\left(1-(1+\frac{1}{n})\mu\right)^{3/2}}$.

Comparison to Schur and Jordan

To compare, note that

$$\frac{1-(1+\frac{1}{n})\mu|\lambda_i|}{\mu-|\lambda_i|+\frac{\mu}{n}} \leq \frac{n}{\mu}(1-\mu^2).$$

i) Jordan:

- If $|\lambda_i| = \mu$ then catch factor $\frac{n}{\mu}$. Hence, Jordan bound is direct corollary.
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Conclude: New bound outperforms Jordan and Schur

Some words about proof

Sufficient to bound $\inf_{g \in W} \left\| z^n + g\, m_{(\mathcal{T}-\mathcal{T}_\infty)} \right\|_W$.

1. Interpolation problem [Nik09]:
   $\inf_{g \in W} \left\| z^n + g\, m_{(\mathcal{T}-\mathcal{T}_\infty)} \right\|_W = \inf_{h \in W} \{ \|h\|_W \mid h(\lambda_i) = \lambda_i^n \}$

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2. Choose good representative: $r\in(0,1)$ and
   $h_r(z) = \sum_k \lambda_k^n \frac{\tilde{B}(rz)}{rz-r\lambda_k}(1-r^2|\lambda_k|^2)\prod_{j\neq k}\frac{1-r^2\bar{\lambda}_j\lambda_k}{r\lambda_k - r\lambda_j}$

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5. Use Spijker Inequality. Let $|\lambda| = (1 + 1/n)\mu$
$\|\mathcal{T}^n - \mathcal{T}_\infty^n\| \leq \sqrt{\frac{1}{1-r^2}} \frac{\mu^{n+1}(|m|+1)e}{nr^{|m|}(1 - r(1+1/n)\mu)} \sup_\lambda \left| \prod_i \frac{1 - \bar{\lambda}_i r^2 \lambda}{\lambda - \lambda_i} \right|$

Conclusions and References

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