Symmetry protection of measurement-based quantum computation in ground states

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Quantum computing can proceed through *measurements* rather than unitary evolution.

Uses a resource state such as the *cluster state*: a universal circuit board.

Resource states can be:
- constructed with unitary gates
- the ground state of a coupled quantum many-body system

Computational properties = properties of states

Raussendorf and Briegel, PRL (2001)
Q: What properties of a state are needed for MBQC?

› There are a handful of proposed resource states

Raussendorf and Briegel

Raussendorf et al.

Wei, Affleck, Raussendorf
Miyake

X. Chen et al.

› Many are PEPS (projected-entangled pair states)
  - tensor network structure allows for the analytic analysis of the effect of measurements
  - natural interpretation as ground states of a local Hamiltonian

› Can we identify the properties of a system that allow for MBQC as a form of robust quantum order?
Symmetry-protected topological order & MBQC

- **Quantum memories**
  - Ground state of the toric code (a local stabilizer Hamiltonian) is a quantum memory
  - Robust to local perturbations
  - Topological order = quantum memory

- **Measurement-based QC**
  - Ground state of the cluster model (a local stabilizer Hamiltonian) is a MBQC resource
  - Robust to symmetric local perturbations
  - Symmetry-protected topological order = MBQC resource

**Symmetry-protected topological order**

- (Restricted form of) topological order protects quantum info
- Symmetry-breaking measurements induce logic gates
- Can indentify families of MBQC resource states even without an analytical description of the ground state
1. 1D spin chain: SPT order implies perfect identity gate with perfect measurements

2. 1D SPT ordered spin chain: far-separated non-trivial (not identity) gates are imperfect, described by a *Markovian* noise model

3. 2D spin chain in a quasi-1D SPT ordered phase: far-separated non-trivial gates are imperfect, described by a *local, Markovian* noise model

By choosing to use this MBQC resource to simulate a fault-tolerant circuit, we have:

**Main Theorem**

- For sufficiently small symmetry-respecting perturbations, the perturbed ground state remains a universal resource for measurement-based quantum computation.

Else, Bartlett, Doherty, NJP (2012)
1D cluster model: the identity gate

Hamiltonian - gapped:

\[ H = -J \sum_i Z_{i-1} X_i Z_{i+1} \]

“Frustration free” (all terms commute):

\[ Z_{i-1} X_i Z_{i+1} |gs\rangle = |gs\rangle \]

Measure / apply local field

Maximally entangled state for teleportation

\[ |\psi^-\rangle \]

Q: What are the essential properties of a qubit wire?
1D cluster model: a symmetry

Hamiltonian - gapped:

\[ H = -J \sum_i Z_{i-1} X_i Z_{i+1} \]

“Frustration free” (all terms commute):

\[ Z_{i-1} X_i Z_{i+1} |gs\rangle = |gs\rangle \]

Hamiltonian possesses a symmetry: \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)

i.e., 2 commuting constants of motion

- (0, 0) \( \rightarrow \) Do nothing
- (0, 1) \( \rightarrow \) Flip red spins
- (1, 0) \( \rightarrow \) Flip blue spins
- (1, 1) \( \rightarrow \) Flip red and blue spins

Four elements:

Do nothing
Flip red spins
Flip blue spins
Flip red and blue spins
Ground state as a tensor network state

tensor network state (matrix product state)

Efficient representations of ground states of 1D gapped systems
Natural language for ground-state quantum computation


Goal:
Characterise properties of tensors, in terms of their symmetry, that make a good qubit wire

3 leg tensor

\[
A^{(i)}_{mn} = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\]

\[i = 1 \ldots 4\]
index for basis of spin pairs

\[m, n = 1, 2\]
‘virtual’ index - contracted
Ground state as a tensor network state

tensor network state (matrix product state)

Cluster model possesses a symmetry: $Z_2 \times Z_2$

Tensors can carry a nontrivial \textit{gauge} representation of this group

For the cluster model, $T_g$ is a projective representation: the Pauli group

\[\begin{align*}
T_{(0,0)} &= I \\
T_{(0,1)} &= X \\
T_{(1,0)} &= Z \\
T_{(1,1)} &= Y
\end{align*}\]
Maximally noncommutative projective representations

Maximally noncommutative:

This projective representation has a special property: a trivial ‘projective centre’

This gives:
1. a unique projective rep
2. an isomorphism between group elements and states in a basis

\[
\begin{align*}
T_{(0,0)} &= I \\
T_{(0,1)} &= X \\
T_{(1,0)} &= Z \\
T_{(1,1)} &= Y
\end{align*}
\]
Good measurement basis

tensor network state (matrix product state)
Symmetry protected topological phases

\[ H = H_0 + \lambda V \]

Symmetry-respecting perturbations alter the ground state, but cannot change the type of representation\(^*\)

Chen, Gu, Wen, PRB (2011)

(*) technically, the second cohomology class
In this case, projective or true representation

Ground state with Pauli gauge representation

Ground state with trivial representation

Phase transition
Decomposition of the MPS bond space

Tensor breaks up into a *structural part* (completely determined by representation) and a *junk* part affected by the perturbation

\[ A(\lambda)^{(g)} = T_g \otimes \tilde{A}(\lambda)^{(g)} \]

Quantum information is encoded in this subsystem:
- properties fixed throughout SPT phase
- SPT order resides here (long-ranged order)

‘Junk’ subsystem describes details of the specific ground state
- decoupled from the qubit
- describes a state with only short-ranged correlations

Singh, Pfeifer, Vidal, PRA (2010)
Ground states in the ‘cluster’ phase possess the long-range entanglement necessary for use as a quantum wire, always with the same special basis measurement.
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Else, Bartlett, Doherty, NJP (2012)
Qubit wires – 1D cluster model

Measure / apply local field

\[ X \ X \ X \ X(\theta) \ X(\phi) \ X \ X \ X \ X \ X \ X \ X \ X \]

‘Rotated’ maximally entangled state for gate teleportation

\[ I \otimes R(\theta, \phi) |\psi^-\rangle \]

information flow

quantum gate
Equivalence to local Markovian error model

\[ A(\lambda)^{(g)} = T_g \otimes \tilde{A}(\lambda)^{(g)} \]

**Problem 1:** adaptive measurements lead to correlations in time

**Solution:** work in a ‘dual picture’ using a nonlocal unitary to remove adaptivity

... ask me later!!
Equivalence to local Markovian error model

Problem 2: nontrivial gates lead to correlations between qubit and junk space
Equivalence to local Markovian error model

**Solution:** separate nontrivial gates beyond correlation length

The ‘junk space’ state serves as an environment

- weak coupling for small perturbations
- noise is local in ‘time’, and independent
- space gates apart beyond correlation length: Markovian noise model
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Else, Schwarz, Bartlett, Doherty, PRL (2012)

Else, Bartlett, Doherty, NJP (2012)
Extension to 2D

2D cluster state through ‘quasi-1D’ model

CZ gates (symmetry-respecting) couple the chains

Extensive symmetry group \((\mathbb{Z}_2 \times \mathbb{Z}_2)^N\)

One realisation through diagonal strips of X-flips
2D cluster model in a nontrivial SP phase

- Ground states in the ‘cluster’ phase possess the long-range entanglement necessary for use as a qubit wire
- Quantum logic gates can be performed, with a local, independent, Markovian error model
- Apply methods of fault-tolerance
Structure of our result

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Else, Schwarz, Bartlett, Doherty, PRL (2012)
Conclusions and future directions

› Ground-state quantum computing requires a type of ‘hidden’ long-range order:
  - symmetry-protected topological order
  - identical to a type of antiferromagnetic order, in some 1D and 2D systems

› How is this order characterised in 2D or higher-D systems?
  - Can we replace our quasi-1D symmetry with a genuine on-site symmetry in 2D?
  - Related to extensions of symmetry-protected order in 2D?

› Can we find systems allowing MBQC with a physically-motivated symmetry group? (e.g., antiferromagnets?)

› Can this order be robust at non-zero temperature?